MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXX. JANUARY 14, 1910. NO. 3.

SIR DAVID GILL, K.C.B., F.R.S., PRESIDENT, in the Chair.

Thomas Lawrence Bennett, M.A., Egyptian Survey Department, Cairo;
William Heathcote Gilfillan, Surveyor-General of the Transvaal, Pretoria;
Rev. William Hall, R.N., Chaplain and Naval Instructor, Whitecote, Paignton, Devon;
Charles Hope Harris, Surveyor-General's Office, Adelaide, South Australia;
James Hunter, Jun., Craigmore House, Craigmillar Park, Edinburgh;
Robert Howard Nolan MacDonald, B.A., Milleen, Dalkey, Co. Dublin, Ireland;
Henry Lloyd Meadows, M.A., Ballyrane, Killinick, Co. Wexford, Ireland;
Bert Perrott, F.C.S., Bryn Clydach, Neath Abbey, Neath, Glamorgan;
Rev. Edmund Sixtus Phillips, Professor of Science, Oscott College, Birmingham;
Abraham Barsley Taylor, 33 Lansdowne Road, Aylestone Park, Leicester;
Hari Prasad Vidyant, M.A., C.E., F.C.S., Executive Engineer, Public Works Department, Makbulganj, Lucknow, India;
James M. Williams, F.R.G.S., F.L.S., Duffryn, Parkstone Avenue, Canford Cliffs, Bournemouth; and
William Matthew Worssell, Assistant, Transvaal Observatory, Johannesburg, S. Africa,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Albert Henry Stuart, B.Sc. Lond., Science Master, Leamington Municipal Schools, 52 Greatheed Road, Leamington Spa (proposed by F. W. Dyson);

Capt. H. E. Purey Cust., R.N., Hydrographer of the Navy (proposed by Sir W. H. M. Christie);

John Edward Austin, West Court, Delting, near Maidstone (proposed by Richard Inwards); and

Evelyn G. Hogg, M.A., Christ's College, Christchurch, New Zealand (proposed by E. F. J. Love).

Eighty presents were announced as having been received since the last meeting, including, amongst others:—

Photograph of a portion of a model of the Moon made by the late M. Stuyvaert, and a portrait of M. Stuyvaert, presented by Madame Lightford.

Astrographic Chart: 23 charts from the Bordeaux Observatory (with volume of the catalogue), 6 charts from the Paris Observatory, and 15 charts from Toulouse Observatory, presented by the French Minister of Public Instruction; 9 charts, presented by the Observatory, Uccle, Belgium; and 25 charts from the Royal Observatory, Greenwich, presented by the Astronomer Royal, completing the Greenwich zone of the chart.

On the Diagrammatic Representation of Proper Motions.

By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. A recent paper by Professor Dyson (*Proc. R.S.E.*, vol. xxix. part iv. No. 21) makes it clear that a considerable number of proper motions can be assigned with tolerable certainty to one or other of the two apices found by Kapteyn and Eddington. Professor Dyson used Mr. W. B. Blaikie's device for solving spherical triangles to deduce the position-angle and distance of the apex from each star, and tabulates these quantities. The method is in every way convenient numerically, especially for those who have the apparatus mentioned. But spherical coordinates do not readily lend themselves to graphical representation; and after working with them under various difficulties for a time, the following method was devised for representing the sphere on the flat so that the directions of the proper motions can be simply and quickly drawn.

2. Project the sphere from its centre on to the circumscribing cube, and then unfold the cube as in fig. 1. Then the following

considerations are perfectly general, and independent of any system of axes.

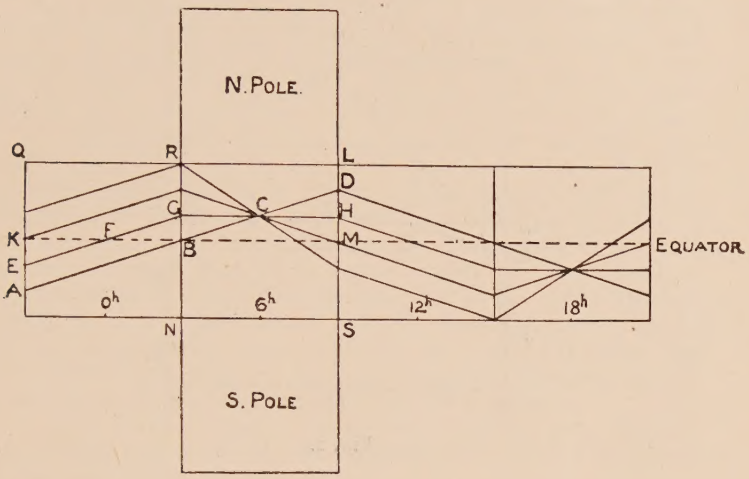


FIG. 1.

(*a*) Great circles of the sphere are represented by a series of straight lines on the faces forming broken lines.

(*b*) The lines KEA and MHD represent meridians 180° apart, of which the points K, E, A correspond to M, H, D respectively. Hence KE = MH, KA = MD. If then we are given any line AB in one square, we can draw it in the adjacent square by cutting off MD = KA and joining BD.

(*c*) All lines through mid points of the side common to two adjacent squares (such as B) are unchanged in direction in the next square.

(*d*) Lines such as EFG which pass through the centre of a square (F) are parallel to the common axis in the next square (GH parallel to BM).

(*e*) Parallel lines in one square such as AB, EG, etc., all meet in a point C on the y axis of the next square; they are, of course, again parallel in the next square, and meet in another point on the y axis of the fourth.

(*f*) These properties of adjacent squares QN and RS are not limited to the actual squares, which may be extended indefinitely in the y direction; *i.e.* QK, RN, LS may be produced indefinitely both ways. Thus in fig. 2, although AB does not cut the side ND, it cuts ND produced in C, and can be continued along CDE, where SE = QA. If the line be such as FG, which would meet ND produced at an inconvenient distance, we may produce the line first in the square QM along GH; then we may suppose this square turned round R as pivot until M falls on L and H on X.

The continuation of FG will then be XY where LY = KF. It is clear that the points XY are given without any inconvenience.

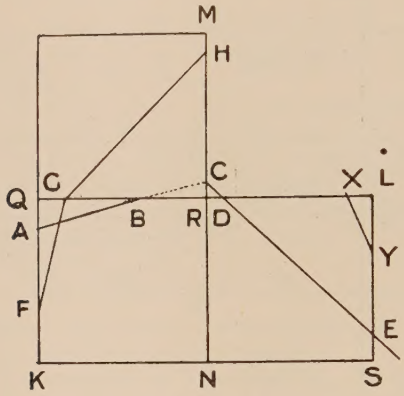


FIG. 2.

3. *To join two points* A, B *in adjacent squares.*—Let A, B be the points in adjacent squares, centres O, Q. The problem resolves itself into finding the point in square Q which corresponds either to A *or to its anti-point.* This can be done by drawing any two convenient lines through A and producing them in square B as in the previous paragraph. Two convenient

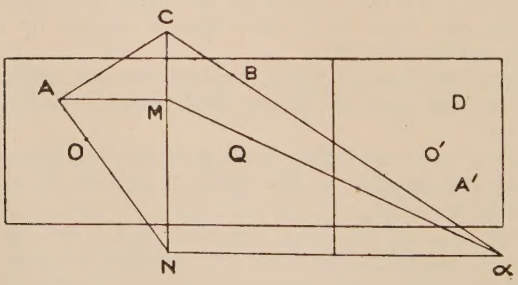


FIG. 3.

lines will often be those joining A to the centre AON, which in the next square is Nα parallel to OQ, and AM parallel to OQ, which in the next square passes through Q. These meet in α, which in this instance corresponds to the anti-point of A. Join αB to cut NM produced in C. Then ACB is the required great circle joining A and B.

4. *To join two points* A, D *in different squares not adjacent.*—It is clear that the above method may be extended by finding

the point corresponding to α in the next consecutive square; there need never be more than two such operations.

But a simpler process is available. A square which is not adjacent to another must be opposite to it, and then we take directly the anti-point A′ in the square O′ and join it to the required point D. This line can then be produced into other squares. Or we may, of course, take the anti-point of D in A's square.

5. *Difficult cases.*—The only cases of difficulty are when both points A and B lie near the centres O, Q of their squares, for then the point α corresponding to A is at a great distance from Q, and the point β at a great distance from O. (If either A or B is not near a centre, we can select that one for transference to the next square.) In such cases it is *nearly* correct to join AB direct. Suppose we do so and it cuts the line CN in K. Then the real position of K is a little further from the line of centres OQ, and can be found without much difficulty by "trial and error," the test of correctness being the equal (and opposite) intercepts of the lines KA and KB on the opposite sides of the squares.

6. But we must have a more exact procedure than this, and one solution (quite possibly not the simplest) is afforded by the use of poles and polars.

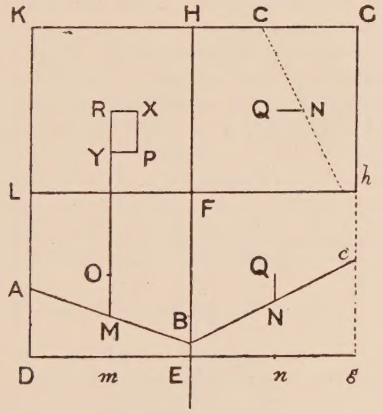


FIG. 4.

Consider a point P, and draw the lines PX, PY parallel to the sides of the square in which it lies. If we take OM = RY in the line RO, it is clear from the similarity of the squares that YM is an arc of 90°. Hence M is the pole of the line PY, and hence lies on the polar of P. If QN = RX, similarly N lies on the polar of P in the square FG. Rotate this (as we are entitled to) into the position F*g*. Then the polar of P is the line joining MN. In this particular case we can find the point B very simply, for

$$BE + AD = 2(Om - OM)$$
$$BE + cg = 2(Qn - NQ).$$

If we denote the side of a square by 2, and remember that $AD + cg = 2$, then

$$BE = 1 - OM - QN = 1 - RY - RX.$$

Thus when P is specified, the points M, B, N are specified at once, and the polar can be drawn. Similarly, if the line AB be given, it can be produced along Bc, and the pole P can be found from OM and QN. If now we have any two points in adjacent squares LE, Fg, draw the polars in the square KF; and then we can draw the polar of the intersection, which is the line joining the original two points.

7. Thus, since we can pass from one square to another, we may limit our attention to the geometry of a single square. The geometry does not stop short at the edges of the square itself, which may be extended to any angular distance from the centre short of 90°. But the distances from the centre are on a scale of tangents, and increase rapidly after about 60°. The corners of a square are at an angular distance θ where $\tan^2\theta = 2\tan^2 45°$, so that $\theta = 55°$ approximately. We may therefore work up to angular distance 60° in what follows.

8. There is no difficulty about plotting points in the polar squares from R.A. and declination by the use of circles and straight lines. The distances from the centre are proportional to tan N.P.D. or cot δ: the angles are straightforward. Hence we are chiefly concerned with the equatorial squares. Let X′OX be the equator, the meridian through any star S projects into the line SM perpendicular to the equator, and if α be the R.A. of a star, A that of the plate centre, then

$$OM = \tan(\alpha - A).$$

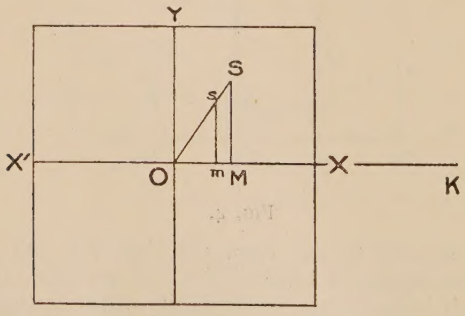


Fig. 5.

With a scale graduated for tangents on which $OX = \tan 45° = 1$, we can set off the length OM without difficulty.

9. The same can be done for the Y coordinate if we find q where

$$\tan q = \tan\delta \sec(\alpha - A).$$

This formula is familiar to those who use photographic "Standard Coordinates." The angle q is the angle OKS on the sphere, where K is the centre of the next square along OX, and therefore distant 90° from O (see fig. 6). If we took coordinates with respect to K, instead of with respect to P, then SN would correspond to

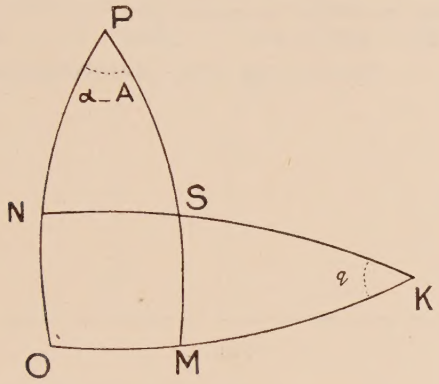


FIG. 6.

declination (SM) and the angle round K (NKO) to right ascension round P (OPM). But the fact is that there is a great convenience in using, not either of these systems alone, but a mixture of both; for the angle OPS round P and the angle OKS round K convert directly into rectangular coordinates. It is a little curious that we should pick out the two R.A. coordinates. One usually thinks of declination as more convenient because it does not change scale. But for our present purpose it seems clear that the above angles (or what amounts to the same thing, the arcs OM and ON) are the most convenient pair.

10. Now in the projection of fig. 5 we have

$$\mathrm{MS} = \tan\delta \sec(\alpha - \mathrm{A})$$

and

$$\mathrm{OM} = \tan(\alpha - \mathrm{A}).$$

Take

$$\mathrm{O}m = \sin(\alpha - \mathrm{A})$$

and draw ms parallel to MS to cut OS in S. Then

$$ms : \mathrm{MS} = \mathrm{O}m : \mathrm{OM} = \sin(\alpha - \mathrm{A}) : \tan(\alpha - \mathrm{A}) = 1 : \sec(\alpha - \mathrm{A})$$

or

$$ms = \tan\delta.$$

11. We may thus use a scale of the kind shown in fig. 7, one edge of which is graduated in sines and the other in tangents.

Since the projection is symmetrical with respect to the point O, this scale can be used for any radius and any transversal. Thus if PR be any straight line (fig. 8), set the side AC of the scale (fig. 7) on PQ, and slide it till AB passes through O. Rule the

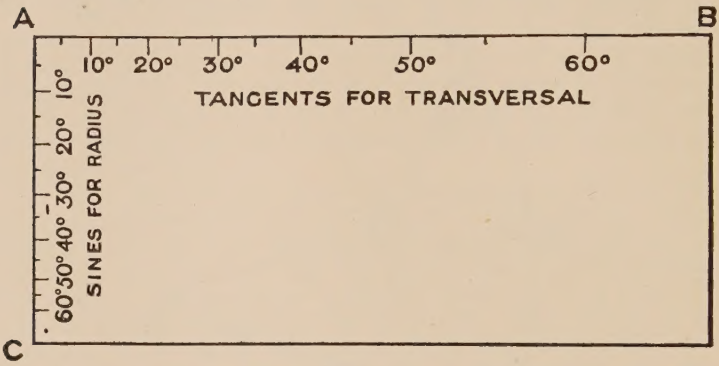


FIG. 7.

perpendicular OQ and read off its length on the scale of tangents, say 25° ; then turn the scale round and set O to 25° on the scale of sines, the point A being now at q ; read off qr on the scale of

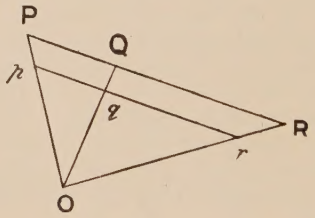


FIG. 8.

tangents, which gives the length of QR on the sphere. Similarly, qp gives the length of QP, and hence we get the whole length PR by three very simple operations.

12. It remains to find the spherical values of the angles at a point—such as the angle OPR, say—and then we can translate any geometrical problem from sphere to plane, and *vice versa*.

Let the angular values on the sphere of the lengths OQ and QP be θ and ϕ. Then, putting in fig. 8,

$$x = \mathrm{OQ} = \tan\theta, \quad y = \mathrm{QP} = \sec\theta \tan\phi,$$

we have

$$dx = \sec^2\theta.d\theta, \quad dy = \sec\theta.\sec^2\phi.d\phi,$$

and in the neighbourhood of Q, $\sec^2\phi = 1$. Hence if a great

circle on the sphere makes an angle S with the arc OQ, and the projected line on the plane makes an angle P with OQ, we have

$$\tan P = \frac{dy}{dx} = \frac{\sec\theta . d\phi}{\sec^2\theta . d\theta} = \cos\theta \tan S.$$

Hence angles from the radius in the plane are smaller than those on the sphere. Their spherical equivalents can be read off at once by the use of a scale similar to that shown in fig. 7 ; for if,

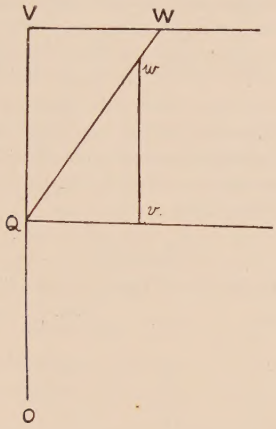


Fig. 9.

in fig. 9, VW is a scale of tangents, so that VW = tan S, and if QV = sec θ, then

$$\tan VQW = \tan S \cos\theta = \tan P.$$

Hence we want a square scale of which the side AB is divided as before (fig. 7) according to tangents, and AC is now divided according to secants. Since sec θ is always greater than unity, the secant scale will begin where the sine scale leaves off, and thus a single scale will suffice if we extend the side AC.

13. When the angle VQW exceeds 45°, it is preferable to use the complementary angle wQv. In this case if $vw = \tan(90° - S)$, we must have $Qv = \cos\theta$. Now a cosine scale is essentially the same as a sine scale if we read off the complementary angle. It might be better to have a separate scale in order to avoid mistakes ; but it will be seen that a single scale would suffice for all the work, one side AB having tangents, and AC sines up to 90° and secants beyond that. Of course it should be graduated on both faces, so that it could be used right or left.

14. It will be seen that the measurement of a length between two points, and the setting out of a position-angle, both require more than one operation in general. For the position-angle we must read off both the angle of the radius from north point, and then the direction of the proper motion. But when once the

scale is provided, the operations are so simple that they do not take long.

15. By such methods the proper motions of faint stars found on Oxford plates for the Astrographic Chart, and tabulated in *Monthly Notices*, lxix. p. 57 and p. 491, were ruled on a diagram, and the apices A and B were indicated. But the position found for A was sensibly different from that adopted by Professor Dyson (6^h, $-10°$); a position at about ($7\frac{1}{2}^h$, $+10°$) suits these particular P.Ms. much better. This will be seen from the following Table I., which presents the results in a different manner by giving in the last two columns the intercepts on the adopted axes—the equator and the meridian of 6^h. It will be clear that the R.A. of the apex is indicated best by the 2nd and 4th groups, where the lines cut the equator at a considerable angle; and the mean value is sensibly greater than 90°. Indeed there is no value $<90°$ except that for the last star but one, which would be bracketed if we take a new value for the R.A. of the apex.

TABLE I.

Oxford P.Ms. of Faint Stars. Apex A (6^h, $-10°$).

Oxford No.	Mag.	R.A.	P.M. Pos. Angle.	Intercepts on Axes. Equator.	Mern. 6^h.
		°	°	°	°
+26, 259	9·2	3	98	(75)	− 8
+26, 355	11·0	3	40	(163)	+53
+26, 3131	9·2	18	79	(131)	+22
+26, 4673	11·2	29	74	(153)	+24
+28, 12579	11·4	88	147	105	(+25)
+26, 20681	9·2	108	180	108	—
+26, 22642	10·9	116	144	133	—
+26, 23412	10·1	120	191	115	—
+26, 23419	6·0	121	180	121	—
+26, 23968	9·2	124	180	124	—
+26, 24255	8·9	124	194	118	—
+26, 24107	10·0	125	185	123	—
+26, 25798	11·2	135	232	105	(−16)
+26, 25941	10·0	135	238	99	(− 8)
+26, 25973	6·0	136	202	127	(−60)
+26, 33210	8·0	199	296	(60)	+20
+26, 34103	10·4	205	284	(86)	+ 7
+26, 34208	8·5	208	300	(61)	+21
+26, 55357	8·0	288	23	98	(+20)
+26, 67901	7·5	310	339	139	(−64)
+26, 74061	8·2	325	61	106	(+10)
+28, 67189	8·9	326	55	111	(+16)
+28, 69159	10·7	335	70	103	(+ 8)
+28, 69669	11·0	338	70	(106)	+ 8
+28, 70256	9·0	340	97	(55)	−16
+26, 78862	8·7	351	45	(147)	+49

From the second group the R.A. comes out 116°, and from the last 108°. If we adopt 112°, and then bracket R.As. determined from stars more than 45° on either side of this meridian, *i.e.* between 157° and 247°, and again between 337° and 67°, then we must bracket the equator intercept for the last three stars. The general mean is now about 115°, or R.A. $7^h\ 40^m$. Using the other stars to get the declination of the apex, we find from the first group + 23°, from the third + 16°, and from the last three stars now bracketed, + 13°. Thus the position of the apex indicated by these observations is

	R.A. $7^h\ 40^m$	Dec. + 17°
as against	R.A. 6 0	Dec. − 10°

adopted by Professor Dyson.

16. It is not intended to put this result forward as a final determination, but merely to draw attention to the considerable discordance from the adopted apex, which can scarcely be accidental if we may judge from the agreement of the individual lines. The discordance was first noticed by the graphical method, but is seen quite as clearly in the figures above tabulated. It does not seem that we can ascribe it to the faintness of the stars, for if we make two solutions as below,

	R.A.	Dec.
11 stars brighter than 9·0	117°	+ 18°
15 stars fainter than 8·9	113	+ 17

the discordance is certainly not *more* marked for the faint stars.

17. Returning to the diagrammatic method, it is not necessary to set out actual position-angles in order to obtain these results.

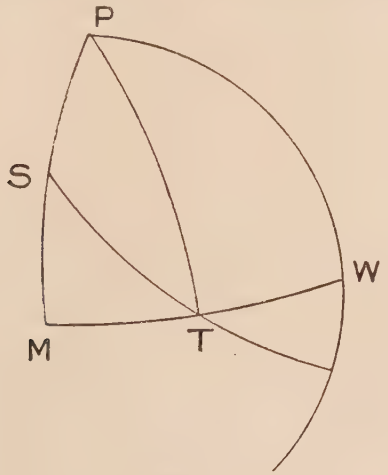


Fig. 10.

In fig. 10 let S be a star moving in the direction ST and let P be the north pole, MT the equator. We are given the angle PST,

the position-angle of the proper motion. And in the triangle SPT, since PT is a right angle,

$$\tan \mathrm{SPT} = -\tan \mathrm{PST}.\cos \mathrm{SP}$$
$$= \tan \mathrm{MST}.\ \sin \delta.$$

Knowing PST and δ, we can obviously find SPT with the scale. If MST exceeds 45°, use the complements of SPT, MST, and δ with the equation

$$\tan (90^\circ - \mathrm{SPT}) = \tan (90^\circ - \mathrm{MST}) \sec (90^\circ - \delta).$$

Hence we can always find the arc MT, and therefore the point T where the line ST cuts the equator, which gives one of the intercepts above mentioned. Then joining ST and producing it if necessary, we find the other. This procedure is, however, not very satisfactory when S and T lie near together, and thus we are led to consider an independent process for the other intercept. Let PK be one of the standard meridians, passing through the centre of a square. Then in the triangle SPK we know the angles at S and P and the side SP, and we require the side PK. The equation connecting the four parts is

$$\cot \mathrm{PK} = \cot \mathrm{SP} \cos \mathrm{SPK} + \sin \mathrm{SPK} \cot \mathrm{PSK} \operatorname{cosec} \mathrm{SP}.$$

18. It is possible to solve this equation by use of the scale, but the solution is rather long, and consists of several successive operations. It is better to choose PK differently. Let it be, for instance, the meridian 90° from PS, which in the plane representation of the sphere is the corresponding meridian in the adjacent square. Then if $\mathrm{SPK} = 90^\circ$ the equation becomes

$$\cot \mathrm{PK} = \cot \mathrm{PSK} \operatorname{cosec} \mathrm{SP}$$

or

$$\tan \mathrm{PK} = \tan \mathrm{PSK} \sin \mathrm{SP}$$

which is of the form negotiable with the scale. Of course we might find the angle q (corresponding to R.A.) and the arc corresponding to declination, and then find the intercept on the y axis precisely as on the x axis, but this involves several operations.

19. Let us now proceed to the apex B stars. There are 12 in $+26^\circ$ zone and 6 in $+28^\circ$ zone, as shown in the following table. The striking thing is, that there are no B stars between $0^h\ 36^m$ and $10^h\ 11^m$. Of course this may be an accident, but there are a dozen A stars between the same limits: we must await further information on this curious point. The 18 stars which have P.Ms. passing near the B apex are shown in Table II., where the value of MT, deduced as above, is given in column 5. Adding it to the R.A. of the star in the 3rd column, we get the "equator point" where the line cuts the equator. But apex B, for which Professor Dyson adopts the position $17^h - 60^\circ$, falls on the south polar square.

TABLE II.

Oxford P.Ms. of Faint Stars. Apex B.

Oxford No.	Mag.	R.A.	Position Angle.	MT.	Equator Point.	Intercepts on Axis. R.A. 270°.	R.A. 0°.
26, 28107	7·5	153°	151°	+13°	166°	+28°	−68°
26, 34238	9·1	208	166	+ 6	214	+15	+22
26, 34290	9·5	209	120	+37	246	(+75)	(+53)
26, 50702	11·5	278	194	− 6	272	+86	−15
26, 50932	11·8	278	201	− 8	270	±90	−16
26,	11·8	287	172	+ 3	290	−22	+ 8
26, 66226	11·0	306	196	− 7	299	+22	−13
26, 66463	9·5	308	241	−38	270	(±90)	(−38)
26, 67632	11·2	311	191	− 4	307	+16	−13
26, 67823	11·6	309	193	− 5	304	+21	−13
26, 68464	11·0	310	209	−13	297	+39	−23
28, 63432	11·1	315	194	− 7	308	+22	−16
26, 74272	9·2	325	243	−40	285	(+83)	(−54)
28, 66910	11·5	326	249	−52	274	(+85)	(−58)
28, 66897	6·8	326	239	−39	287	(+78)	(−55)
28, 67027	9·3	326	239	−39	287	(+78)	(−55)
28, 69875	10·8	338	207	−13	325	+34	−44
28, 1540	4·1	9	222	−24	345	+41	−73

It is therefore convenient to have the intercepts on the axes of this square, which are the lines from the South Pole towards the R.As. 0°, 90°, or (if we please) 180° and 270°. Since the apex B lies near the 270° meridian, let us take this as positive and tabulate the distances from the South Pole in which it is cut as in the 7th column. The 8th column gives the intercepts on the perpendicular meridian, counting distances towards 0° as positive.

20. Now a glance at the results seems to indicate two groups rather than one; the results bracketed seem to belong rather to an apex much nearer the equator. If we exclude them, then the diagram shows that the others all pass very near the apex $17^h - 60°$. To find the best position of the apex we might drop perpendiculars on to all the lines and find the minimum sum of their squares. Representing by θ and ϕ the angle-intercepts given in the last two

columns, the linear intercepts are $\tan\theta$ and $\tan\phi$; and the equation to any line is

$$x\cot\theta + y\cot\phi - 1 = 0,$$

so that the perpendicular from (x, y) is

$$p = (x\cot\theta + y\cot\phi - 1)/(\cot^2\theta + \cot^2\phi)^{\frac{1}{2}}.$$

If we make Σp^2 a minimum we have

$$\Sigma\cot\theta(x\cot\theta + y\cot\phi - 1)/(\cot^2\theta + \cot^2\phi) = 0$$

and $$\Sigma\cot\phi(\quad,,\quad)/(\quad,,\quad) = 0$$

from which x and y can be deduced. But the present material seems scarcely sufficient to justify a solution on these lines.

21. There remain a number of stars giving lines which do not pass very close to A or B. Two other apices C and D are suggested, and inquiry was made of Professor Dyson whether there was any support for them among the stars he had examined. He very kindly examined his exceptional cases, and found a certain amount of support for apex D (near $12^h\ 0°$). But more material is required.

22. Meanwhile it seems worth while publishing these notes since—

(*a*) They suggest that the proper motions obtained, especially those of faint stars, indicate apices other than those at present adopted. The position of apex A seems to be sensibly different; and apex B is possibly not simple but compound.

(*b*) To elucidate such points, the graphical methods here described may be found useful. It seems clear that we have before us a great deal of work in the study of proper motions in this way, and graphical methods will save time by indicating the main features of distribution, which can then be investigated by more accurate numerical methods.

[*Note added February 2.*—At the moment of going to press it was noticed that it would have been much simpler to use the polar square throughout; or rather the polar square extended to include declination 26°. Position-angles with the meridians are then set out with great facility; indeed it is not even necessary to calculate the position-angles, for the lines can be drawn from the Δx and Δy measured on the plates. When a line is once drawn, its intersections with any convenient meridians can be noted and tabulated. Its equator-point is indicated by the meridian to which it is parallel. The whole procedure is quite elementary.]

Radial Movement in Sun-spots. (*Second Paper.*)

By J. Evershed.

In the *Monthly Notices* for March 1909 I gave a preliminary account of some observations of line-displacements in the spectra of sun-spots,* and showed that the systematic nature of these displacements and their dependence on the position-angle of the spectrograph slit could be simply explained on the hypothesis of an outward radial movement parallel to the Sun's surface of the gases in the reversing layer over spots. It was necessary also to suppose that the movement accelerated outwards from the umbræ of spots, attaining a maximum value of about 2 km. per second at the outer edges of the penumbra. I have now to record the results of further investigations of the movements in spots which have been carried out during the past year.

At the outset it may be stated that the spectrum of every considerable spot appearing during the year, or reappearing after one or more rotations of the Sun, has been photographed at some time during its transit across the disc, and, without exception, all have shown the same systematic line-displacements. The lines have invariably been found to incline towards the red on the side of the spot nearest the Sun's limb, and towards the violet on the side nearest the centre of the disc, whatever position the spot may have occupied on the disc, if not too near the centre. In a few instances the motion has been detected even when the spot was on or very near to the central meridian.

If the displacements of the lines are really due to motion in the line of sight, as there seems no reason to doubt, the conclusion is, I think, inevitable that the above-mentioned hypothesis represents the truth, and that all spots are centres of a force directed radially outwards in a horizontal plane, acting continuously on the materials of the reversing layer over the entire area of a spot, and maintained throughout the life of a spot.†

* See also *Kodaikánal Observatory Bulletin*, No. 15.

† In a paper read before the Royal Academy of Sciences, Amsterdam, Professor Julius attempts to explain the systematic line-shifts in spot spectra by anomalous dispersion. In several important particulars, however, the facts I have observed do not conform with the consequences of this theory.

Professor Julius states that when the slit bisects the spot in the direction of the centre of the disc the Fraunhofer lines must be more or less curved in the shape of the letter S. This I have never observed, except in the case of the central absorption-lines of H and K. All the finer lines of the spectrum are quite straight, although inclined, over spots. Also the great majority of the lines are as sharply defined at their edges over the penumbræ where the displacement is greatest as on the general surface of the Sun, being displaced bodily from their normal positions. Anomalous dispersion would, however, cause a marked widening and indefiniteness at the edges, a spreading towards the violet on the side nearest the centre of the disc, and towards the red on the limb side.

Professor Julius' fundamental assumption that the gases above the photospheric level have densities sufficient to give appreciable refraction effects seems to me to be, on several grounds, highly improbable.

An interesting question which arises in connection with this radial movement is—Where is the source of supply? Where do the gases come from which are for months at a time continuously radiated away from the umbræ? Considering the most probable source to be the interior regions immediately beneath the reversing layer, or perhaps below the photosphere, I have attempted to discover vertical motion in the umbræ of spots, a welling up of gases from below. An upward movement at all comparable with the radial motion would be easily detected by line-displacements which, unlike those due to horizontal movements, would attain a maximum value for spots at the centre of the disc, diminishing with linear distance from the centre.

It is at once evident from a mere inspection of spectrum photographs of spots near the centre of the disc that this vertical motion is in all cases very small in comparison with the horizontal motion. A good series of spectra obtained specially with a view to determining vertical motion has not yet been secured, but some measures have been made of the lines in five different spot umbræ more or less well situated as regards the centre of the disc. No motion of ascent could, however, be detected in any of these; on the contrary, after correcting the measures for small displacements which I believe to be due to pressure effects, a small residual displacement was obtained towards the red, indicating a movement of descent in the umbræ of the order of 0·4 km. per second (details of these measures are given in *Memoirs of the Kodaikánal Observatory*, vol. 1, part i. If this result should be confirmed by future measures, and a motion of descent found to be general in spots, the inference would be that the source of supply was in the higher chromosphere, perhaps entirely outside the Sun. This would favour the views expressed by Mr. Newall in his recent Presidential address.*

On the other hand, it is perhaps unnecessary to assume any appreciable ascending motion, considering the probable enormous increase of density with the depth of the gases at the level of the photosphere. A practically unlimited quantity of the rarefied vapours of the reversing layer might be supplied from the denser regions a very short distance below, without any apparent movement upwards.

Radial Motion in the Higher Chromosphere.—I have said that the spectrum-lines are invariably found to be displaced towards the red on the side of the penumbræ nearest the limb, and to the violet on the opposite side, thus indicating an outward movement. There are, however, two notable exceptions among the host of absorption-lines in the solar spectrum. The central absorption-lines known as H_3 and K_3 (possibly also the underlying emission-lines H_2 and K_2), when clearly defined in the penumbræ, are displaced in exactly the opposite sense in every case I have examined, that is, towards the violet on the limb side of the spot, thus indicating an indraught of the calcium vapour of the higher chromosphere.

* *Monthly Notices*, lxix. 333.

In the following table I give a few measures of relative motion on opposite sides of the spots, showing that this inward movement is of the same order of magnitude as the outward movement of the lower gases.* The lines H_3 and K_3 represent, as is well known, a region extending roughly between 3″ and 10″ above the photosphere, whilst the bulk of the Fraunhofer lines showing the outward movement are produced in a region not exceeding 3″ in height.

TABLE I.

H and K in Spots, with Slit in Direction of Centre of Disc.

Date.	Central Distance. Radius = 1.	Approximate Latitude.	Relative Displacements in Å on opposite sides of Spot.		Deduced Horizontal Velocities. Mean of H and K.
1909.		°	H.	K.	Km. per sec.
Apr. 1	·68 E.	+ 9	0·037	0·039	2·1
7	·77 W.	− 10	0·029	0·040	1·7
Sept. 10	·60 W.	− 2	0·043	0·016	1·8
14	·98 W.	− 3	0·059	0·051	2·1
Nov. 11	·91 W.	− 7	0·047	0·024	1·5
14	·65 E.	− 4	0·043	0·046	2·6
,,	·71 E.	− 7	0·025	0·016	1·1
30	·73 W.	− 14	0·040	0·032	1·8
,,	·52 W.	− 13	0·023	0·028	1·8

Average horizontal velocity = 1·83 km. per sec. on each side of a spot near outer limits of penumbra.

The direction of motion is in all cases one of approach on the side nearest the limb, and of recession on the side nearest the centre of the disc, and the greatest motion is found near the outer edge of the penumbræ.

Now it is interesting to note that whilst the lines of the reversing layer appear perfectly straight, but inclined, where they cross a spot, there being an appreciable break or jolt in the lines at the points where they pass from the penumbræ on to the surrounding photosphere, this is not the case with H and K, which generally form regular sinuous curves over spots when the slit lies in the direction of the Sun's centre. With the slit in the opposite direction they, in common with the other absorption-lines, remain perfectly straight and undisplaced provided that no eruptive disturbance is in progress. The following diagram (fig. 1) will give a general idea (much exaggerated) of the behaviour of H and K as contrasted with a line of iron when the slit is in the direction of the centre, or radial at the limb.

* In the table it will be noticed that H and K in some cases give discordant values of the displacement. This is due to the difficulties of measurement; frequently the narrow absorption-lines H_3 and K_3 are absent on one side of the spot, and the settings have then to be made on the much more diffuse emission-lines H_2 and K_2.

An explanation of this contrasted behaviour of the Ca and Fe lines is suggested by Professor Michie Smith: it is that the low-lying metallic vapours in their motion outwards penetrate into or even perhaps beneath the banked-up faculæ surrounding the spot, so that there is an apparently sudden stoppage of the motion at the limits of the penumbra. The calcium vapour of the higher region, on the other hand, in its movement inwards towards the umbra, meets with no such obstruction; the motion can therefore be traced from a considerable distance outside the penumbra.

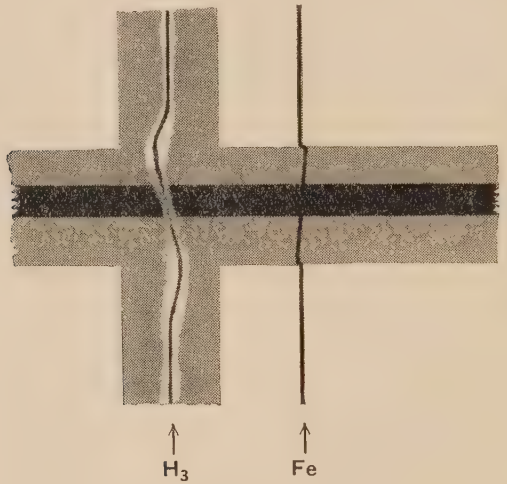


FIG. 1.

The inclination of the calcium lines crossing the spot would seem to indicate a diminishing velocity towards the umbra, just as the opposite inclination of the other lines indicates an accelerated movement. This, however, cannot be affirmed until it has been shown that there is no vertical motion in the calcium vapour. It would seem natural to suppose that the calcium vapour is drawn downwards as well as inwards, and I cannot at present say whether this is so or not, although some of my measures indicate such a downdraught.

Spiral Motion of the Outflowing Gases.—In the preliminary account of these researches already referred to it was stated that the greatest line-displacements were found when the spectrograph slit bisected a spot in the direction of the centre of the disc; and that if the Sun's image were turned through 90°, so that the slit was parallel to the tangent at the nearer limb, the displacements disappeared. This nodal position of the slit implies an outward movement in straight lines from the umbræ, whereas any slight displacement of the nodes from 90° would mean that, superposed upon the radial movement, there was a relatively slow rotation.

Combining the two movements, there would result a spiral motion, the gases flowing outwards in more or less curved lines.

The examination of good spectroheliograph images, and visual observation of the radial filaments in the penumbræ of spots, suggested that in many cases at any rate the motion *is* in curved lines, if these filaments represent, as I think they must do, the actual stream-lines of the moving matter. To test the matter, I have selected from our collection of photographs a few spectra for measurement in which the following conditions have been fulfilled:—

1. Spectrograph slit exactly normal to direction of centre of disc.
2. Spot not less than half way from the centre towards the limb.
3. Atmospheric conditions good.
4. Scale of spectra not less than 1 mm. = Å.

If there should be a displacement of the line of nodes of several degrees in either direction from the normal (tangent at the limb), minute but measurable displacements of the lines on each side of a spot should be found.

The method adopted in measuring the plates was in most cases similar to that used for determining pressure shifts in spot spectra (described in Memoir No. 1 of the Kodaikánal Observatory), using a sliding screen placed over the negative to limit the field of view alternately to separate parts of the image, and using a single thread placed parallel to the spectrum lines In two spectra it was found advantageous to measure the double displacements resulting from the superposition of a pair of positive copies of each negative by a laborious method, which, however, gives smaller probable errors than the direct method.

In all the spectra measured, unmistakable evidence of small displacements of the lines at the north and south edges of the penumbræ was found, indicating a displacement of the line of nodes amounting in the mean to about 8°.

I give in Table II. the results of these measures, expressed in km. per sec. I have assumed that the observed motion is the component in the line of sight of a motion parallel to the Sun's surface. My justification for this rests, it must be admitted, on very meagre negative evidence, having so far found no appreciable vertical motion in the penumbræ when situated near the centre of the disc. The observed velocities may be found by multiplying the tabulated velocities by the figure given in column 2, which is the sine of the angular distance of the spot from the centre of the disc. The observed velocities are always a little smaller, according to the distance of the spot from the limb, where the component in the line of sight of a horizontal movement becomes unity.

[Table.

TABLE II.

Slit normal to Direction of Sun's Centre.

Date.	Central Distance. Radius = 1.	Latitude. Degrees.	Horizontal Motion. Km. per sec. N.	S.
1909 March 6	·81 W.	− 9	−0·32	+0·32
,, ,, ,,	·57 W.	+ 3	+0·37	−0·37
,, April 7	·77 W.	−10	−0·17	+0·49
,, July 28	·80 W.	+ 7	+0·32	−0·32
,, Sept. 11	·78 W.	− 2	−0·64	+0·24
		Means	0·36	0·35

Note.—The values obtained in the first and second determinations (March 6th) are the mean movements of the north and south edges of the penumbræ. The method of measuring does not in these cases admit of separate estimates at the two sides of the spots except as regards sign. The same remarks apply also to the measurement of radial motion on July 28th given in Table III.

The velocities found are all of the same order of magnitude, and indicate an average movement of about $\frac{1}{3}$ of a kilometre per second on each side of a spot. The measures were made at the outer limits of the penumbra in each case, where the displacement of the lines is greatest.

This movement, found at the extremities of a diameter normal to the direction of the Sun's centre, and roughly north and south, probably indicates a rotation of the gases at the outer edges of the penumbræ. But it may mean simply an easterly or westerly drift on the north and south sides of a spot. The question can only be decided by means of similar measures made on spots situated near the central meridian, when a normal to the direction of the centre will bisect the spot in an east and west direction. If the motion at the extremities of an east and west diameter were found to be the same as at the north and south, a circulating movement could be affirmed. Unfortunately, the displacement of the lines is too minute to be detected except with a spot situated more than half way from the centre towards the limb.

The movement measured cannot be due to the relative motion which would be acquired by particles moving radially from the spot in a north and south direction (analogous to the deflection of the trade winds on the Earth), for this could not exceed 0·015 km. per sec. after travelling over 1° of solar latitude, or a distance about equal to the radius of a moderate-sized spot. Moreover, the motion would be opposite in direction to that actually found; that is to say, for a north spot there would be a westerly drift on the north side, and an easterly drift on the south side.

This would become, for a spot approaching the west limb, a recession on the north side and an approach on the south side of the spot, relative, of course, to the surrounding undisturbed surface of the Sun. On referring to Table II., it is at once seen that the two northern spots give a + movement and the three southern spots a − movement on their northern sides.

If we assume a rotational movement, the change of signs

revealed in the measures, in passing from a northern to a southern spot, is interesting as indicating an opposite direction of rotation in the two hemispheres, the direction being clockwise in the south.

For comparison with the above, I give in Table III. the horizontal radial velocities of the same spots, deduced from measures of spectra taken with the slit in the direction of the centre of the disc. Here we have no change of sign with change of hemisphere, but always a motion of approach on the east and recession on the west sides of spots nearing the west limb, that is, of approach on the side nearest the centre of the disc. This is, of course, the effect of the radial movement; and proof that it is in all directions from the spot has been obtained by observing the displacements in spots on the central meridian, and in other situations on the disc.

TABLE III.

Slit in Direction of Sun's Centre.

Date.	Central Distance. Radius = 1.	Latitude. Degrees.	Horizontal Motion. Km. per sec. E.	W.
1909 March 4	·43 W.	− 9	+3·09	−3·50
,, ,, 6	·57 W.	+ 3	+2·60	−1·93
,, April 7	·77 W.	−10	+1·26	−2·34
,, July 28	·80 W.	+ 7	+1·30	−1·30
,, Sept. 10	·60 W.	− 2	+1·10	−2·08
		Means	+1·87	−2·23

Summarising these results, we have a mean rotational movement of 0·35 km. per sec., and a radial motion of about 2 km. per sec. Combining them, we get a spiral or catharine-wheel movement opposite in the two hemispheres. This is represented graphically in the appended diagram (fig. 2), which shows the directions of the stream-lines in a northern and in a southern spot.

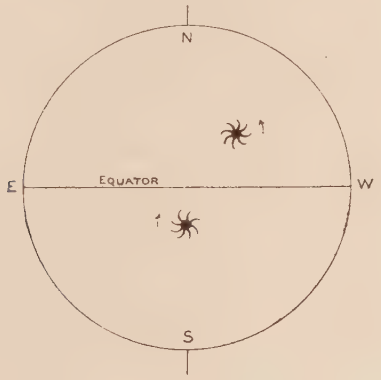


FIG. 2.

It might be supposed that a spiral movement such as that here illustrated would betray itself in the disposition of the calcium

flocculi over spots. It is to be remarked, however, that the circular movements I have described are confined entirely within the penumbral limits, and no motion has yet been detected outside those limits. Also it appears to be confined to the lower chromosphere. In good spectroheliograph plates of the calcium flocculi the regions of the penumbræ in quiet spots often show radial stream-lines, and these are sometimes curved. In general, however, the more brilliant masses of flocculi of the surrounding region show no tendency to either a radial or a spiral structure, although the outlying portions frequently break up into circular or horseshoe formations, which are seen in an attenuated form all over the disc. But the minute structure of these rings is always granular, and gives no suggestion of a whirling motion.

It was at first rather baffling to find that these spirals are reversed in direction compared with the stream-lines shown in some of Hale's H_α photographs. This reversal is, however, readily explained if we assume that the hydrogen represented in Hale's plates partakes of the indraught which I have shown occurs in the higher chromosphere from the behaviour of the lines H and K. If the inflowing gases of the higher regions and the outflowing gases of the reversing layer all partake of a slow rotational movement, then we should have oppositely curved spirals in the two regions.

I shall not discuss at length the bearing of these results on the Zeeman phenomena of spots, mainly because my measures as regards rotation are preliminary only, and will probably be modified by future research; I cannot say whether the movement is universal in spots. The apparently radial structure, without curvature, in the penumbral filaments, often seen in symmetrical spots, suggests a purely radial movement in many cases.

The opposite movement in the two hemispheres, although clearly shown in five instances, may not be an invariable rule. Referring to the magnetic field in spots, Professor Hale states that "Although the larger spots in the northern and southern hemispheres of the Sun are usually found to be of opposite polarity, it frequently happens that spots of opposite polarity occur in the same hemisphere, sometimes in the same spot group." * One would expect, therefore, if the rotation movement bears a causal relation to the magnetic field, to find occasional instances of spots showing a clockwise rotation in the north, or *vice versa*.

It appears to be extremely doubtful whether this apparent rotational movement will, by assuming an ionised condition of the moving gases, be found adequate to explain the magnetic field. So far as my observations go, the motion seems always to be greatest at the outer edges of the penumbræ, and it becomes quite imperceptible in the umbral region; yet the Zeeman broadening of the lines is, I think, invariably greatest over the umbræ, and decreases to an inappreciable amount at the outer edges of the penumbræ.

* *Publications of the Astronomical Society of the Pacific*, xxi. 205.

It is perhaps more probable that the characteristic movements (radial and rotational) taking place in the penumbræ are secondary effects, and that the electric vortex is entirely beneath the reversing layer. Professor Hale has found that the magnetic field decreases rapidly with height above the photospheric level:—

"The doublets and triplets of iron give the strongest fields hitherto measured in sun-spots. The D lines of sodium and the *b* lines of magnesium, which are produced at a higher level in the solar atmosphere, indicate a much weaker field. The hydrogen line over sun-spots, representing a still higher level, give no indication of a magnetic field." *

There is a remarkable and perhaps significant analogy in this with the radial-motion effect. I have found that the greatest motion of 2 to 3 km. per sec. is found in the region of iron absorption. The line D_1 of sodium is scarcely, if at all, affected on the few plates I have taken of this region of the spectrum (D_2 is apparently unaffected, but an interfering atmospheric line at 5890·10 will perhaps account for this).† The *b* lines of magnesium are affected more than D_1, but much less than the neighbouring iron lines. Hydrogen, if affected at all, seems to show traces of an opposite movement, as is shown by the lines H and K of the higher chromosphere.

It may be well to state here that in measuring the displacements due to radial motion I have usually selected lines having an intensity not greater than 4 on Rowland's scale. These are mostly iron lines, but include also the lines of Co, Ni, Cr, V, and Ti. Most of the stronger lines appear to give much smaller values of the motion, and there is a tendency among the fainter lines to give a rather wide range of values, even among lines of the same element. Probably the movement varies enormously with level, and the values I have given in Tables II. and III. must be taken to represent the region of greatest motion.

A thorough investigation of the differences found among the different lines of an element, as well as between those of the different elements, is very desirable; but progress in this direction has been slow, partly owing to the numerous other lines of inquiry opened up, and also to the laborious nature of the measuring work.

* *Loc. cit.*, page 207

† The sodium lines at 5682·869 and 5688·436 give much larger values of the radial motion than D_1, but not so large as the average of the iron lines measured.

The Moon in Ultra-violet Light: Spectro-selenography. By R. W. Wood, Professor of Experimental Physics, Johns Hopkins University, Baltimore. (Plate 9.)

(*Communicated by Professor Sir Robert Ball.*)

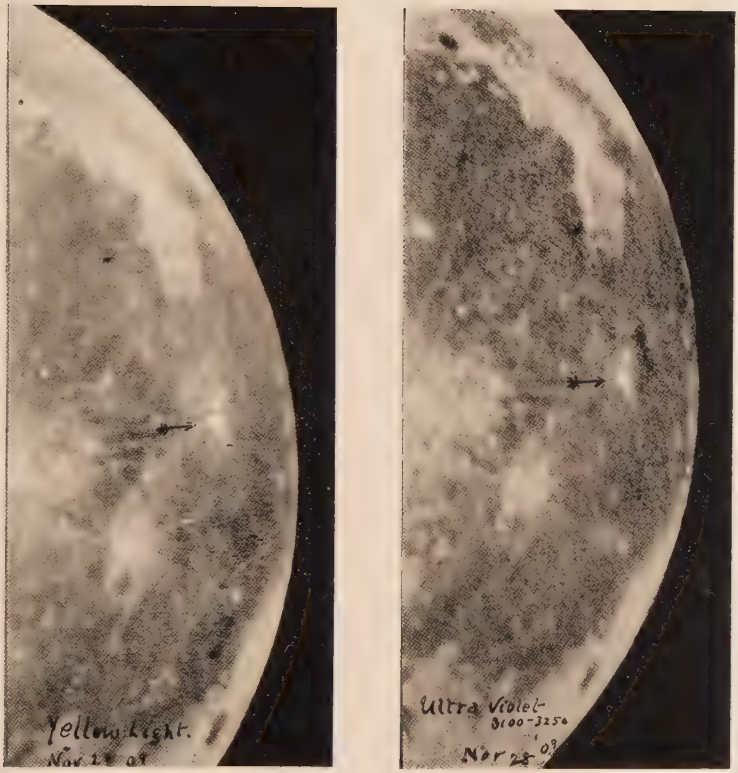
In the present communication I propose to show the possibility of undertaking a study of the petrography of the Moon's surface by means of photography with light confined to comparatively narrow ranges of the spectrum, utilising both the visible rays, the ultra-violet, and even the infra-red, at least the portion lying between wave-lengths 7000 and 7600, which plays very little part in ordinary vision.

I have already discovered an extensive deposit of some material which is dark in ultra-violet light, close to the crater Aristarchus. This deposit shows scarcely at all in the pictures made in yellow light, while it is nearly black in pictures made by light confined to the range 3100–3250. Light of this wave-length will not pass through glass, and it was necessary to use a lens of quartz, coated with a film of metallic silver, which absorbed the visible and a large portion of the ultra-violet rays. The focal length of the lens was 6 ft. and its aperture 3 in.

The preliminary experiments were made at my summer laboratory at East Hampton, Long Island, N.Y., with an improvised instrument made out of odds and ends. The equatorial mounting was made of an old bicycle frame imbedded in a block of cement, the steering axis pointing to the pole. A slow motion enabled me to make exposures of several minutes if necessary. A more detailed description of this instrument will be found in the *English Mechanic* for November 12, 1909.

Parallel experiments made in the laboratory showed that many substances which are white in ordinary light are jet black when photographed with these very short waves. Chinese white (zinc oxide) and most white garden flowers are good examples. These white flowers, if growing on a snow-bank, would be nearly invisible, and would not appear in photographs made in the usual way, but would be clearly brought out in pictures made with the quartz lens and the silver film.

This illustrates the principle which I am applying to the study of the Moon's surface. A full account of these experiments, both with the ultra-violet and infra-red rays, will appear shortly in the *Century Magazine*. Landscapes made with the infra-red radiations are most extraordinary, for the sky is as black as night and the foliage snow-white. Since my return to Baltimore I have repeated the experiments on the Moon with a concave metal speculum of 6 in. aperture and 11 ft. focus, mounted on the equatorial telescope of the University. It was not possible to obtain very sharp photographs, on account of the jars to which the instrument is subjected.



THE MOON IN YELLOW AND ULTRA-VIOLET LIGHT, SHOWING ARISTARCHUS DEPOSIT.—R. W. WOOD.

(It is mounted at the top of the tower of the physical laboratory, and there is much heavy traffic at all hours.)

The pictures, however, show indisputable evidence of a vast deposit of some substance, not found elsewhere upon the Moon's surface, to one side of the crater Aristarchus. It is almost invisible in the pictures made with yellow light (aurantia screen and isochromatic plate). Its appearance is shown on Plate 9, the crater Aristarchus being indicated by an arrow. This is the only deposit that I have been able to find, thus far, which is thus clearly differentiated in ultra-violet light, though it is probable that a longer-focus mirror with better facilities for accurate following would show other deposits of smaller area. There is a region a little above Aristarchus, near the Moon's limb, in which the light and shade is differently distributed in the two photographs, though the differentiation is not as marked as at the Aristarchus deposit. Examination of photographs of the full moon taken with various instruments shows that this deposit appears on photographs made without a colour screen on ordinary plates, while it is nearly invisible on isochromatic plates made with a yellow screen. With my small reflector, however, I get very little trace of it except with the quartz plate and silver film. A nitrose screen, which transmits another range of the ultra-violet (3400–3700), shows it, however, though much less distinctly.

In the study of various volcanic ejecta and igneous rocks, as they appear when photographed in yellow and in ultra-violet light, I have found two specimens of volcanic tuff, which appear exactly alike in yellow light, while in ultra-violet (3100–3250) one appears very much darker than the other. This same differentiation is shown in the case of sulphur and zinc oxide. The fact that the bright streaks on the Moon come out equally bright in the ultra-violet pictures proves that the white substance is neither sulphur nor oxide of zinc.

The work that has been done thus far is only to be regarded as preliminary. I have not yet made any pictures with the extreme red end of the spectrum, and only two or three with the nitrose screen.

From now on, the work should be done with larger and more perfect instruments. Silver on glass reflectors will not do, for silver only reflects about 4 per cent. of the light with which we have to deal. A metal speculum of 30 or 40 ft. focus would be about right, and I hope to hear of some observatory or amateur who is equipped for and ready to go on with the work. The preparation of the silver film on the quartz plate requires great care, as deposits made in the usual way are usually not of uniform density, and therefore show structure by transmitted light. It is my plan to build a long-focus horizontal reflector, if no instrument exists suitable for the work. The work will prove very interesting, but if it can be done elsewhere I do not feel justified in going to the expense of constructing an instrument especially for it. I shall be glad to hear from anyone who will volunteer to take up the

work. If I hear nothing before April 1st, I shall shoulder the responsibility myself.

I shall be glad to supply directions for preparing the ray filters, and can furnish the quartz plate if necessary. An exposure of about 1·5 minute is necessary with a 6-in. mirror of 11 ft. focus.

A Last Word on the Correlation of Variable Stars.

By Karl Pearson, F.R.S.

I do not think the controversy with Mr. Plummer can really be carried further with profit to the readers of the *Monthly Notices*. What I should like to ask those who have studied this controversy is, having read Mr. Plummer's *Second Reply*, to carefully re-read my previous *Rejoinder*, vol. lxix. pp. 573–85, and judge how far I have met or not met already his standpoint in this second paper, for I see no substantial change in his arguments.

Our difference may be very briefly summed up in what I term an essential divergence with regard to the rule of primary and deduced variables. Thus the statement he makes on p. 7, "Remembering that statistics, as such, are concerned merely with numbers, and not things," essentially differentiates our positions. I hold that statistics are only of meaning as far as the things they measure are kept constantly before us. Thus when he places

$$\left.\begin{matrix} \mathrm{R} \\ \mathrm{D} \end{matrix}\right\} \quad \text{against} \quad \left.\begin{matrix} \mathrm{L} \\ \mathrm{D}-\mathrm{F} \end{matrix}\right\} \quad , \quad \left.\begin{matrix} \mathrm{R}-\mathrm{L} \\ \mathrm{F} \end{matrix}\right\}$$

I cannot follow him, because, from my standpoint, R and L are independent primary variables—separate organs, the right and left femur, having independent origin—and R – L is a derived variable having no separate existence. On the other hand, dog and flea are primary variables, and D – F is a derived variable having no independent existence, and the comparison between R – L and F from my standpoint cannot be made. It is precisely the same with the Algol stars; I cannot lose sight of the things which the numbers represent. m_2 and m_1 are not primary variables at all. m_2, representing maximum light, is independent of the secondary body and is a primary variable; m_1, representing a portion of this light, is not a primary variable; the light, however, cut off by the secondary depends on a new set of causes, the size and orbit of the secondary, and is a primary variable. There is no obvious physical reason whatever why this should be related to the other primary variable, and any relationship discovered must be due (i) to common distance, (ii) to a relationship between light emitted by primary and the size and orbit of secondary. It has been the custom of certain anthropometers to measure H_2 the height of a man standing, and H_1 his height sitting. It is clear that there will always be a correlation between these two measurements, as they contain a common factor, the upper body-length. From these measurements the anthropometer deduces

$H_2 - H_1$ the length (approximately) of the lower limbs. The correlation of $H_2 - H_1$ and H_1 is *organic* because it is a relationship between two parts, the upper and lower body, with sensibly no common element, although algebraically H_1 appears in both. The correlation of H_2 and H_1 is spurious, because they have a common part. This is the case of Mr. Plummer's D and D – F; it is not the case of my right and left femur, nor, from *my* standpoint, is it the case of the Algol stars. Mr. Plummer says, "let all the bonds to which we are attaching importance be broken by completely deranging the order of the two series" (p. 9)—*i.e.* put height sitting H_1 of one man with height standing H_2 of a different man, and we shall still find $H_2 - H_1$ correlated with H_1. I agree entirely; we have introduced spurious correlations, because the bonds we have broken are not those between our primary variables. But the argument does not show that the correlation between $H_2 - H_1$ and H_1, when we pair the parts of the same man, is not wholly organic and important, for $H_2 - H_1$ and H_2 are the primary variables—the organs with spatial differentiation which can be considered as having, and in the dissecting-room are given, separate existence. It is only this fundamental difference between our attitudes—that the statistician must and does study the things which his numbers represent: that, I think, is worth referring to once more, otherwise I am well content to leave other minor matters to the judgment of the future. I have little doubt that when astronomers in that future use the calculus of correlation, as they now use the integral calculus, as an unquestioned instrument of investigation, they will to some extent be inclined to smile over the difficulties which apparently Mr. Plummer and I find in coming to any common basis for discussion.

The Long Period Variable RT Cygni (Ch. 7085) in 1909.

By A. N. Brown, M.A. (Plate 10.)

The last of my observations published in *Monthly Notices*, vol. lxix., No. 7, and made on 1909 February 7, with those preceding it, indicated that the light of this Variable was steadily declining. The 75 light-estimates which follow give my results for the remainder of 1909. As in the previous series, the dates of maximum and minimum are assumed to be those upon which a smooth curve drawn through the observations reaches the highest and lowest points respectively. Two minima and one maximum are shown on the curve appended.

It is noticeable that, while the descent from the maximum which occurred on 1909 January 2 to the minimum of 1909 April 24 took 112 days, the descents immediately preceding and following respectively occupied 104 and 103 days; and, again, that the Variable only took 83 days to rise from the minimum of 1908 October 11 to the maximum of 1909 January 2, whereas the

ascent from minimum to maximum on the occasions immediately preceding and following this occupied 88 and 87 days respectively.

Referring to the observations, the date is given in the 1st column below, and in the 2nd the last four figures of the Julian date corresponding. In the 3rd column the letters T, t, and p denote respectively refractors of 5-in., 3-in., and 2-in. aperture, the latter telescope only having been used when it was impossible to employ the others. Following each of these letters is the magnifying power used. The 4th column shows the value attached to each observation, 1 signifying good, reliable; 2 fairly good—the Moon, clouds, etc. having caused a certain amount of difficulty. The comparisons in the 5th column were for the most part made by the "fractional" method, but each is here expressed in steps, the value of a step in every case being ·1 magnitude. All the comparison stars used are numbered as in Hagen's Atlas, the negative sign after the comparison star implying that the Variable is fainter, the positive sign that it is brighter, than the star in question. In the last column the magnitude deduced from the comparisons is given, a mean being taken when two or more separate estimates have been made. As regards the magnitude of each comparison star, the Harvard value, where this was forthcoming, has been adopted; when the H.C.O. does not assign a value to one of Hagen's stars used, Hagen's own value, as given in the *Atlas Stell. Var.*, has been taken. In the curve a dot shows a 1st class, a cross a 2nd class observation.

The Observations.

Date.	Julian date.	Instr.	Class.	Comparisons.	Ded. mag.
1909.					
Feb. 9	8347	t. 30	1	7 − 2, 10 + 1·5	8·2
19	8357	T. 50	,,	10 − 2, 14 + 1	8·6
21	8359	,,	,,	7 − 5·5, 14 + 1; 10 − 1, 13 + 0·5	8·5
Apr. 7	8404	T. 200	,,	43 − 3, 44 + 1	10·9
20	8417	t. 150	,,	43 − 3, 44 + 1	10·9
22	8419	,,	,,	44 − 1	11·1
24	8421	,,	,,	44 − 1	11·1
25	8422	,,	,,	44 − 1	11·1
May 1	8428	,,	,,	43 − 3, 44 + 1	10·9
4	8431	T. 150	,,	43 − 3, 44 + 1	10·9
6	8433	,,	,,	43 − 2·5, 44 + 1·5	10·9
7	8434	,,	,,	43 − 2, 44 + 2	10·8
8	8435	T. 200	,,	43 − 2·5, 44 + 2	10·8
10	8437	T. 150	,,	41 − 1·5, 43 + 0·5	10·6
11	8438	T. 167	,,	40 − 1·5, 43 + 0·5	10·5
18	8445	T. 50	,,	30 − 2, 41 + 1·5	10·2

The Observations—continued.

Date. 1909.	Julian date.	Instr.	Class.	Comparisons.	Ded. mag.
May 19	8446	T. 50	1	30 − 1, 41 + 3	10·1
21	8448	,,	,,	25 − 2·5, 30 + 1, 27 − 2	10·0
23	8450	,,	,,	23 − 1·5, 25 + 0·5	9·7
25	8452	,,	,,	23 − 1·5, 26 + 1, 25 − 0·5, 27 + 1	9·7
31	8458	,,	,,	15 − 4, 20 + 1, 16 − 4·5, 19 + 1	9·2
June 8	8466	,,	,,	14 − 1, 15 + 0·5	8·8
10	8468	t. 75	,,	14 − 0·5, 15 + 1, 16 − 1, 19 + 5	8·8
18	8476	T. 50	,,	5 − 1·5, 7 + 0·5	8·0
27	8485	,,	,,	5 − 1, 7 + 1·5	7·9
29	8487	t. 30	,,	4 − 1·5, 5 + 0·5, 4 − 1·5, 6 + 1	7·8
30	8488	,,	,,	4 − 0·5, 6 + 2, 4 − 1, 5 + 3	7·7
July 1	8489	,,	,,	3 − 8·5, 4 + 1	7·5
July 2	8490	T. 25	1	3 − 7·5, 4 + 2·5	7·4
4	8492	t. 30	,,	3 − 7, 4 + 3	7·4
8	8496	,,	,,	3 − 7, 4 + 3	7·4
10	8498	,,	,,	3 − 5, 4 + 5	7·2
17	8505	,,	,,	3 − 7·5, 4 + 2·5	7·4
18	8506	,,	,,	3 − 6·5, 4 + 3	7·3
19	8507	,,	,,	3 − 6·5, 4 + 3	7·3
20	8508	,,	,,	3 − 5, 4 + 5	7·2
23	8511	,,	,,	3 − 7, 4 + 3	7·4
24	8512	,,	,,	3 − 5, 4 + 5	7·2
26	8514	,,	,,	3 − 7·5, 4 + 2	7·4
28	8516	,,	,,	3 − 8, 4 + 2	7·5
Aug. 5	8524	p. 16	,,	3 − 7·5, 4 + 2	7·4
13	8532	p. 40	,,	4 − 1·5, 6 + 1	7·8
19	8538	p. 16	,,	=6, 7 + 0·5	7·9
23	8542	p. 40	2	=7, 10 + 1	8·2
Sept. 2	8552	t. 30	1	10 − 1, 13 + 0·5, =12	8·5
6	8556	t. 60	,,	15 − 4, 20 + 1, 19 + 1	9·2
8	8558	,,	,,	19 − 0·5, 22 + 1, 20 + 1	9·3
11	8561	t. 30	,,	14 − 7, 22 + 1, 19 + 1, 20 + 1	9·3
14	8564	T. 50	,,	19 − 1, 22 + 1, 22 + 1, 25 + 3	9·4
18	8568	T. 120	,,	22 − 1, 27 + 2, 25 − 1	9·7
19	8569	,,	,,	25 − 1	9·8
Oct. 5	8585	,,	,,	43 − 2·5, 44 + 2	10·8
8	8588	T. 150	,,	43 − 3, 44 + 1·5	10·9
17	8597	T. 167	,,	43 − 3, 44 + 1	10·9
18	8598	,,	,,	43 − 3, 44 + 1	10·9

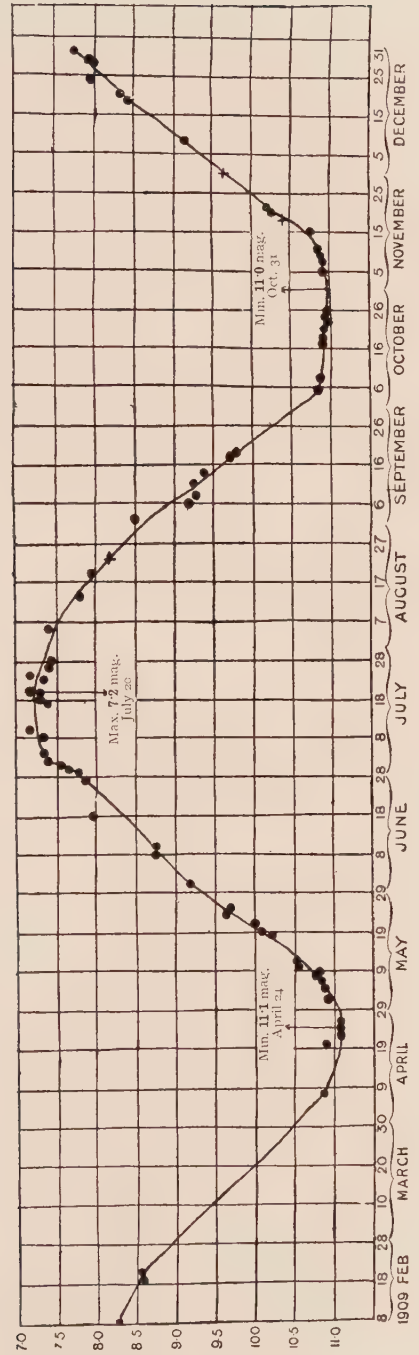
The Observations—continued.

Date.	Julian date.	Instr.	Class.	Comparisons.	Ded. mag.
1909.					
Oct. 21	8601	T. 200	1	43 − 3, 44 + 0·5	10·9
23	8603	T. 150	2	43 − 3·5, 44 + 0·5	11·0
24	8604	T. 167	1	43 − 3, 44 + 0·5	10·9
25	8605	T. 200	,,	43 − 3·5, 44 + 0·5	11·0
Nov. 5	8616	T. 167	,,	43 − 3, 44 + 1	10·9
7	8618	T. 200	,,	43 − 3, 44 + 0·5	10·9
8	8619	,,	,,	43 − 3, 44 + 1	10·9
10	8621	T. 167	,,	43 − 2·5, 44 + 2	10·8
15	8626	T. 200	,,	43 − 1·5, 44 + 2·5	10·8
18	8629	T. 150	2	43 + 2	10·4
20	8631	T. 95	1	30 − 2·5, 41 + 1	10·3
21	8632	,,	,,	30 − 2, 41 + 2	10·2
30	8641	t. 75	2	22 − 1, 41 + 7·5	9·6
Dec. 8	8649	T. 50	1	15 − 3, 20 + 2, 16 − 4·5, 19 + 1·5	9·1
18	8659	t. 30	,,	7 − 4, 12 + 1, 10 − 1, 13 + 0·5	8·4
20	8661	T. 50	,,	5 − 6, 13 + 1, 7 − 2·5, 10 + 2	8·3
24	8665	t. 30	,,	5 − 1·5, 7 + 0·5	8·0
28	8669	,,	,,	5 − 2, 10 + 4	8·0
29	8670	,,	,,	5 − 0·5, 7 + 1, 5 − 1·5, 10 + 4·5	7·9
31	8672	,,	,,	4 − 1·5, 6 + 1, 5 + 1	7·8

The Comparison Stars used.

Hagen's number.	Magnitude. H.P.	Magnitude. Hagen.	Hagen's number.	Magnitude. H.P.	Magnitude. Hagen.
3	6·67		20		9·3
4	7·64		22		9·5
5		7·8	23		9·5
6		7·9	25		9·7
7		8·0	26		9·7
10	8·39		27	9·84	
12		8·5	30		10·0
13	8·54		40		10·4
14	8·67		41		10·4
15		8·8	43		10·6
16	8·71		44		11·0
19		9·3			

Ludgrove, New Barnet, Herts:
1910 *January* 10.



LIGHT CURVE OF RT CYGNI IN 1909.—A. N. BROWN.

Note on the General Aspect of the Principal Markings on the Planet Jupiter during the Apparition of 1907–8.

By Scriven Bolton. (Plate 11.)

Observations of the planet Jupiter during the apparition of 1907–8 extended over a period of more than ten months, from 1907 August 27 to 1908 July 4 inclusive, the opposition of the planet occurring in the meantime on 1908 January 29. The general appearance of the principal surface features is here exclusively dealt with. The widths of the belts have been derived from the measures for the determination of latitude, while the colour estimates, unless otherwise stated, represent a mean of several observations extending throughout the apparition.

Two Newtonian reflectors were employed, one, by Calver and clock-driven, having an aperture of $18\frac{1}{4}$ inches, and the other, a With-Browning, of $10\frac{1}{4}$ inches aperture.

N. N. N. Temperate Belt.—A number of faint dark spots and irregularities were exhibited here. To the south the belt bordered several white oval spots situated in the N. N. temperate zone, while to the north it formed a well-defined boundary to the dark polar area.

N. N. Temperate Belt.—On the average, this belt was a trifle darker than the corresponding band of the other hemisphere, but was decidedly fainter than the S. temperate belt. In the first part of the apparition it appeared both of moderate width and intensity. Later, however, it grew somewhat feeble in places, and its insignificance in certain regions called for special remark On the whole, it presented normal activity, and was continuous throughout. Its average width, which was fairly uniform, was $2^{\circ}\cdot6$. It exhibited a cool colour, which, under favourable seeing, was estimated to be steely-grey. A number of faint dark spots were visible along the belt, and these were associated with wispy shadings extending along the N. N. temperate zone.

N. Temperate Belt.—Several noteworthy outbreaks of dark matter occurred in this latitude. The belt was well defined, and was narrow, considering the activity which it displayed. Dark matter appeared to burst forth in many regions, more especially on the red-spot side of the planet, and this extended considerably south of the S. edge of the belt. These outbreaks not unfrequently occurred in the same longitudes as the neighbouring dark spots on the N. edge of the N. equatorial belt. As the apparition advanced, the belt grew slightly darker. It varied considerably, however, in intensity, and its continuity was at times questionable. Its mean intensity proved to be a shade darker than the S. temperate belt, while its average width was about $2^{\circ}\cdot0$. In contrast to the activity shown, the belt exhibited a grey hue.

N. Tropical Zone.—Occasionally there could be seen traces of the mid-tropic band which became visible during the disturbances

of this region in 1906. The zone was faintly shaded, and a trifle less white than the S. tropical zone. The southern half, where it fringed the N. equatorial belt, was deep yellow in colour, and at times ruddy. On a few occasions the entire zone presented a decided yellow hue.

N. Equatorial Belt.—The great activity exhibited by this object during the previous apparition had now practically ceased. The belt showed a gradual diminution in activity throughout the period of observation, and the irregularities along both edges became less obvious. It was almost as dark as in 1906–7, and by far the darkest belt on the planet. During the first few months it still retained a percentage of the distinctive brick-red colour which characterised it in the previous apparition, but later on this was replaced by a reddish-brown hue. The colour of this region was more obvious than that of any other. The average width of the belt was found to be 4°·3. Along the N. edge of the belt were to be seen several conspicuous dark spots projecting considerably into the N. tropical zone, and some of them were detached from the belt itself. The S. edge also presented a number of dark crests to a series of shallow bays, many of them being connected to the S. equatorial spots by equatorial wisps mentioned below.

S. Equatorial Belt.—The general aspect of the belt was much the same as in previous years. Its mean width measured 9°·9; its colour, beside that of the N. equatorial belt, was comparatively cool. In different longitudes it was noted as being dull purple-brown and greyish-purple, and slate-brown preceding the hollow. The colour was more obvious and warmer about the *f.* shoulder, as well as along the N. component, than elsewhere. The belt was divided by a light rift, which became more conspicuous as the apparition advanced. Previous to the conjunction of the S. tropic dark area with the red spot, the S. component exhibited both its usual narrowness and faintness for some distance preceding the hollow. The south edge was slightly wavy, the crests of these waves being a trifle darker than the general run of the belt, and from them issued curved wisps, which terminated in the S. temperate belt. The north component, which was the darker of the two, possessed a large number of spots of varying intensity. Twenty-four of these constituted part of a uniform arrangement of equatorial markings, the regularity of which could be detected by systematic observation. The spots in question were usually connected by transverse wisps to a similar number of N. equatorial crests. Some of these crests, however, did not move in harmony either with their neighbours or with the S. equatorial spots. Such independent motion caused the adjacent wisps to lie at varying angles to the equator. The other wisps, on the average, crossed the equator at an angle of about 45°, and intersected each other at the equatorial belt. In nearly all regions the uniformity was at times temporarily affected by local disturbances, and accordingly its existence became more evident from transit estimates of the

individual objects extending over a considerable period than from a single night's observation.

S. Temperate Belt.—For many years this belt has exhibited but little variation either in width or in intensity. It was appreciably fainter than the N. temperate belt, but much darker than the N. N. temperate belt, while its width, as nearly as possible, was $3^{\circ}{\cdot}2$. It invariably presented a brown colour, and was unusually dark and broad over the red spot during the months that the S. tropic dark area was in the act of overtaking that object. In the longitudes of the area, the belt was slightly displaced northwards, and was narrower than elsewhere. In other regions there were a number of irregularities along the S. edge. It was quite continuous throughout, with no trace of duplicity.

S. S. Temperate Belt.—On account of its wide variation in intensity in different longitudes, this band was somewhat difficult to perceive satisfactorily. Its colour was estimated to be steely-grey. The N. edge exhibited many faint condensations. Immediately to the south there was a fainter belt of more slender proportions. It is not improbable that this latter constituted part of the belt in question, which would then appear to have been duplex, the two components being separated by an unusually wide rift.

S. S. S. Temperate Belt.—The dark south polar shading ended abruptly here.

Red Spot and the Hollow.—Prior to the conjunction of the area with the hollow, the red spot presented an appearance not unlike that in former years, and, although very faint, its contour could occasionally be well seen. The central portion appeared as white as the zone in which it was situated, the dark circumference only rendering the spot visible. When the conjunction referred to was in progress, the spot became more difficult to perceive, and at times it appeared to be temporarily effaced. The hollow, on the other hand, maintained its well-defined outline, while the *p.* shoulder was almost as dark as the *f.* one. At and about mid-conjunction the S. edge of the S. equatorial belt preceding the hollow was much distorted as far as the area extended westwards. The *p.* shoulder also was peculiarly shaped and did not attain a height equal to the *f.* shoulder. Shortly before the area arrived at the red spot, the two wisps, which are frequently seen to form an arch over the hollow, grew decidedly darker, though the spot itself failed to exhibit signs of renewed activity.

The South Tropic Dark Area.—This remarkable object may have been likened to a darkened section of the S. tropical zone, confined solidly within two prominent and well-defined wisps. The *p.* wisp, which was usually the darker, issued from a slight projection on the S. equatorial belt, and after bending in the *S.p.* direction, terminated in a dark spot on the S. temperate belt. The *f.* wisp branched out from the same belt, but crossed the zone in the *S.f.* direction. There were several fainter wisps situated within the dark matter comprising the area, and these also issued from

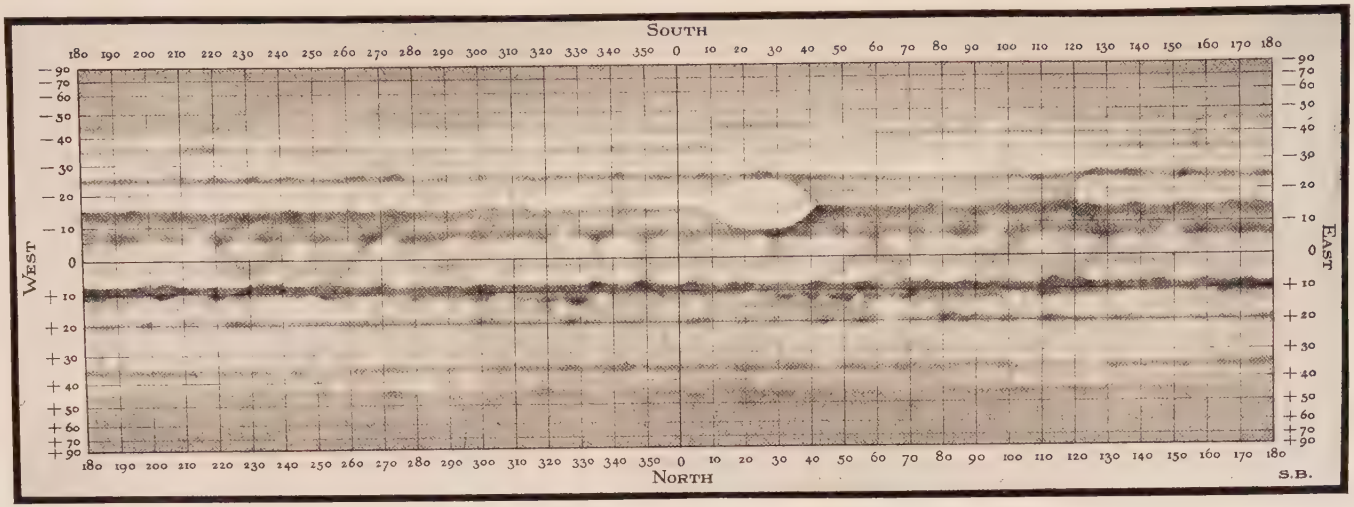
slight projections or crests along the edge of the great belt. Two white spots, one preceding, the other following the area, and which were brighter than the majority of the S. tropical spots, have been objects of interest since 1902. They presented an irregular fluctuation in lustre.

The conjunction of the area with the red spot constituted the most noteworthy phenomenon exhibited by Jupiter during the apparition. As the area approached the environs of the red spot its length exhibited a rapid increase, which increase was maintained until a greater part of the dark matter had appeared on the preceding side of the spot. This was followed by a gradual decrease in its length; had the observations been continued over a longer period than was the case, the area, by the time it wholly preceded the spot, would probably have been seen to be reduced to normal dimensions. Such a phenomenon was witnessed at the conjunction of 1904. In the present instance, the red spot presumably caused the dark matter, which appeared to exhibit considerable elasticity, to move hurriedly past it. This accelerated motion occurred in that part of the area which happened to be approaching conjunction, first the preceding portion, then the middle part, and finally the following end. The strip of the S. temperate belt overlying the area also participated in this sectional movement, which was due chiefly to the increased dimensions referred to. The variation in length may be shown by the figures appended.

Date.	Mean λ of *p.* end at S. Equatorial Belt.	Diminution in λ.	Mean λ of *f.* end at S. Equatorial Belt.	Diminution in λ.	Mean length at S. Equatorial Belt.	Number of transits utilised.	Remarks.
	°	°	°	°	°		
1907.							
Sep.	116·3		187·3		71·0	4	
Oct.	104·5	11·8	169·2	18·1	64·7	6	
Nov.	82·2	22·3	151·2	18·0	69·0	7	
Dec.	71·0	11·2	140·8	10·4	69·8	17	
1908.							
Jan.	54·3	16·7	123·5	17·3	69·2	6	*f.* end in conjunction with red spot, January 16 to March 9.
Feb.			115·6	7·9			
Mar.	2·0	52·3	100·5	15·1	98·5	17	
Apr.	349·5	12·5	90·4	10·1	100·9	8	
May	327·8	21·7	73·4	17·0	105·6	11	Mid-conjunction, May 14.
June	309·5	18·3	53·4	20·0	103·9	3	

The observations showed that, during conjunction, the wisp, which has invariably constituted a well-marked boundary to the *p.*

The PLANET JUPITER during the apparition of 1907-8. Chart of markings observed between 1907 August 27 and 1908 July 4, Inclusive



Longitudes given for date of opposition (1908 January 29). The adopted zero meridian between latitudes -10° and $+8\frac{1}{2}^{\circ}$ is that of System I., and elsewhere that of System II.

end of the area, drifted across or in front of the red spot. Immediately on overtaking the dark arm which connects the following shoulder of the hollow to the S. temperate belt the said wisp was rendered exceedingly faint, and it is believed that at one period it was invisible. When well on its way across the red spot it regained some of its former distinctness, and it was observed to pass the centre of the hollow, and subsequently to approach the *p.* shoulder. Its progress was watched with reference to certain dark spots in a section of the S. temperate belt in similar longitudes to the wisp in question, and all drifted at pretty nearly the same rate. As the wisp drifted without the arch over the hollow, which is formed by the wisps issuing from the shoulders, there was an instantaneous revival in its intensity. It soon became quite clear of the arch, and the intervening space was occupied by the dark matter, the intensity of which was evidently being restored immediately on overtaking the red spot. The wisp was accompanied in its journey by a white spot, which would appear to be identical with the one which has invariably preceded it.

Throughout the epoch of conjunction, the outline of the red spot was more or less obscured by patches of both light and dark matter, which to all appearances scurried across it. The southern half of the spot was perceptibly shaded, this shading being deeper in tone towards the S. temperate belt.

Micrometrical Measures of Double Stars (Seventh Series) and New Double Stars. By the Rev. T. E. Espin, M.A.

The stars measured in 1909 with the 17¼-in. consist of those of Sir John Herschel not previously measured, and various stars occurring in the zones between +45° and +55°.

h's *Stars.*

No.	R.A. 1880.	Decl.	P.	D.	Mags.	Nights.	Date. 1909.
	h m	° ′	°	″			
2538	10 30·8	+44 45	157·1	16·01	9·0 11·5	1	·312
2539	10 30·9	+44 46	59·9	23 05	9·5 13·5	1	·312
2538, 2539			42·1	47·48		1	·312
1348*	18 43·9	+45 58	257·0	6·90	9·6 10·5	2	·789
1382*	19 13·4	+47 48	312·8	7·55	8·5 9·0	3	·671
1386*	15·7	+47 48	325·3	10·23	9·5 9·6	3	·777
1392	18·5	+46 13	235·0	7·17	11·0 11·5	2	·686
1393	19·4	+47 9	119·3	10·27	9·8 9·9	2	·790
1400*	22·3	+45 41	180·8	5·75	10·3 11·0	2	·802
1401	22·7	+47 9	188·8	15·92	9·5 10·2	2	·783
1404*	19 25·2	+46 3	125·6	5·42	8·5 9·0	2	·718 AB
			246·7	17·70	13·0	2	·718 AC

h's *Stars*—continued.

No.	R.A. 1880.		Decl.		P.	D.	Mags.		Nights.	Date. 1909.	
	h	m	°	′	°	″					
1513	20	19·8	+46	8	327·0	13·89	8·5	9·2	3	·816	
1514		20·7	+45	5	216·2	11·69	9·3	10·7	2	·878	
1518		23·0	+45	15	86·2	14·91	9·8	10·0	3	·880	
1549	20	32·2	+47	20	57·0	4·94	9·6	10·5	4	·810	
1624*	21	7·8	+48	7	188·1	5·80	8·5	9·2	2	·677	
1625		8·6	+47	50	80·3	10·05	9·3	9·5	2	·733	
1629		11·2	+46	8	72·4	11·17	8·7	9·2	2	·799	
1633*		13·2	+47	32	59·4	10·97	9·1	9·5	3	·803	
1635*		14·3	+47	17	260·2	11·23	9·0	10·5	4	·824	
1643		19·6	+48	55	22·9	24·19	8·8	11·0	2	·817	
1644		20·3	+47	30	118·5	23·90	9·0	10·0	2	·807	
1645		20·8	+49	43	38·5	7·65	8·5	9·5	2	·817	
1674		34·7	+49	7	327 6	11·87	9·2	11·3	3	·774	
1684		37·4	+49	55	319·4	11·10	8·9	10·1	3	·774	
1710	21	55·0	+50	1	281·2	12·35	8·9	9·6	3	·743	
1732	22	2·8	+49	49	244·8	8·87	9·1	9·6	2	·829	
1777		25·9	+47	49	319·4	11·50	8·5	10·5	2	·773	
1787*		29·1	+47	53	293·3	8·27	9·3	9·4	2	·773	AB
					248·5	10·00	C=	14·0	2	·773	AC
1805		39·6	+46	22	166·9	7·30	9·6	10·1	2	·820	
1812	22	42·6	+46	53	51·7	12·90	9·2	9·8	3	·817	
1855	23	6·3	+44	56	290·0	3·67	9·5	9·6	3	·810	

Various Stars.

Name.	R.A. 1880.		Decl.		P.	D.	Mags.		Nights.	Date. 1909.	
	h	m	°	′	°	″					
B.D. +49° 17	0	4·1	+49	12	293·8	77·29	7·8	7·8	2	·839	
Hu 503		4·5	+49	17	32·9	4·32	8·2	9·9	3	·809	
B.D. +48° 30		4·9	+48	53	310·4	66·66	8·0	8·2	2	·839	
η Cassiopeiæ	0	41·7	+57	11	240·1	5·91			4	·749	
Hu 522	1	12·8	+52	2	88 7	4·06	8·0	12·3	3	·957	
A.G. (Harvard)		24·9	+54	7	197·0	16·26	8·7	8·9	2	·867	
A.G. (Harvard)	1	25·6	+53	39	28·9	14·92	9·0	9·1	2	·867	
Σ 236	2	9·3	+51	55	255·9	1·27	8·5	8·8	3	·968	AB
					326·6	11·58		13·7	3	·968	AC (New)
Arg. VI. 8		42·4	+44	59	340·6	8·97	10·5	11·0	3	·056	BC
					101·3	34·60	A=	9·3	2	·036	AB
A.G. (Harvard)	2	57·8	+50	32	278·7	11·43	9·0	9·1	3	·054	
Arg. VI. 9	3	16·6	+43	51	197·6	26·59	8·7	9·5	2	·112	
Arg. VI. 10	3	24·4	+45	9	165·1	27·68	9·2	9·9	2	·125	

Various Stars—continued.

Name.	R.A. 1880. h	m	Decl. °	′	P. °	D. ″	Mags.		Nights.	Date. 1909.
Σ 446 (Espin 12)	3	40·4	+52	17	251·9	8·53	7·0	9·0	3	·980 AB
					38·9	12·20	C=	11·5	3	·980 AC
					232·5	2·57	10·5	10·9	3	·980 DE
					336·4	66·54			2	·972 AD
Σ 498	4	3·4	+53	28	165·8	1·52	8·9	9·1	3	·906
Espin 707		3·7	+53	26	156·5	9·00	8·1	11·9	3	·881 AB
					287·1	20·24		11·8	2	·893 AC
A.G. (Harvard)		6·3	+50	32	284·4	45·97	8·5	8·9	2	·071
A.G. (Harvard)		13·2	+51	31	25·2	21·69	8·9	9·1	3	·056
Σ 522		13·3	+51	19	32·0	1·72	8·5	8·7	4	·067
Küstner 23		20·6	+55	14	132·3	4·67	9·0	9·7	2	·037
B.D. +52° 843		23·9	+52	33	129·0	47·81	6·7	8·6	3	·980
B.D. +52° 848		24·6	+52	46	356·3	17·55	8·4	8·7	3	·980 AB
					231·9	17·38	C=	11·0	3	·980 AC
					276·4	32·30	D=	10·2	3	·980 AD
A.G. (Harvard)	4	53·4	+51	52	292·2	49·15	8·9	9·0	2	·486
R. Aurigæ	5	7·6	+53	27	239·0	48·03	var.	8·6	2	·504
Σ 1242	8	27·6	+47	32	168·7	2·43	8·6	8·9	3	·187
Σ 1289	8	46·7	+44	3	2·7	3·97	7·7	8·0	3	·187
Σ 1358	9	23·1	+45	12	162·8	23·94	7·0	8·7	2	·141
Σ 1394	9	49·6	+46	29	244·5	3·75	8·3	8·9	2	·226
Σ 1425	10	14·2	+46	45	357·3	4·60			3	·194
Küstner 35		24·9	+48	0	16·0	1·20	9·0	9·2	2	·227
Σ 1467	10	38·2	+45	36	288·8	3·70	8·0	10·0	1	·142
OΣ 374 rej.	19	27·8	+49	57	296·7	18·27	7·1	10·7	3	·787
Küstner 58	20	1·2	+49	18	367·3	2·67	9·0	9·1	2	·787 AB
					159·2	13·88	C=	13·7	3	·791 AC (New)
β 669		26·3	+48	33	341·2	17·45	5·0	13·5	2	·798 AB
					86·2	55·22	C=	11·0	2	·798 AC
H*h* 694		27·2	+48	48	121·2	21·42	6·5	13·5	2	·798 AB (New)
					278·1	60·39			2	·798 AC
Arg. 39 (Espin 92)	20	38·7	+48	50	113·3	9·17	8·2	8·4	3	·599 AB
					126·1	31·20	C=	12·5	3	·599 BC
					233·4	33·42	D=	13·5	2	·595 BD
					56·3	30·15	E=	15·0	1	·605 BE
					257·5	30·67	*a*	14·0	2	·595 A*a*
					345·2	52·05	*b* =	13·2	2	·596 A*b*
63 Cygni (Espin 32)	21	2·5	+47	10	154·7	15·67	4·0	11·7	3	·689
A 788	23	18·7	+45	7	266·4	4·22	8·5	11·9	3	·931 AB
					53·1	11·99	C=	14·0	3	·975 AC (New)
Hu 394		34·4	+49	13	67·3	2·48	9·0	11·0	3	·931
B.D. +49° 4188		35·6	+49	26	243·6	12·97	8·8	9·3	2	·964
A 797		49·3	+46	31	37·8	4·25	8·9	12·0	3	·830
Hu 501		59·3	+49	51	253·4	4·68	7·3	10·8	3	·978
Hu 502	23	59·8	+48	57	110·9	2·72	7·2	10·5	4	·914

Notes.—h's Stars.

h 1348 *h*'s angle 195°. It lies at P 292°·3 D 70″·9 from B.D. +45°2771, and is thus in *h*'s place. If his angle is correct there must be considerable motion.

h 1382 . This is B.D. +47°2797, *h*'s magnitudes are much underrated.

h 1386 In General Catalogue of Double Stars for +45° Decl. read +47°.

h 1400 5ˢ·6 P, 18″·7 S of B.D. +45°2896.

h 1404 C is not given by *h*.

h 1624 This is B.D. +48°3305.

h 1633 „ B.D. +47°3358.

h 1635 „ B.D. +47°3363.

h 1787 C is not given by *h*.

Various Stars.

Σ 236. The new companion is difficult to measure.

Σ 446. The little pair DE has not been noticed before. The *comes* C (Espin 12), I find, was noted at Birr Castle as far back as 1856.

H*h* 694. The nearer companion does not seem to have been observed before.

Arg. 39 (Espin 92). I have taken advantage of the resilvering of the mirrors to measure several other *comites*.

63 Cygni (Espin 32). The *comes* has been quite easy all the season. I have suspected it to be unsteady in light.

A 788. This and several other of the pairs discovered at the Lick Observatory which have faint *comites* have been measured to test the scale of my own magnitudes. Aitken gives the *comes* as 14·2. It was seen at once in bright moonlight on November 29. On December 18 the new *comes* was detected, and measured on this and two other nights.

New Double Stars.

No.	B.D.	R.A. 1900. h	m	Decl. °	′	P. °	D. ″	Mags.		Nights.	Date. 1909.
744	+49° 5	0	3·5	+49	33	33·1	4·52	8·5	11·8	2	·811
745	+49° 10		4·3	49	43	177·2	3·00	9·1	9·3	2	·839
746	+50° 15		5·2	51	2	108·6	6·88	9·2	11·5	3	·889
747	+49° 21		5·6	50	1	174·4	2·90	9·4	10·2	2	·839
748	+51° 23		8·0	51	17	358·6	10·46	7·9	12·0	2	·984
749	+50° 48		12·5	50	45	165·5	9·27	9·1	9·1	2	·842
750			18·2	51	28	265·8	2·70	9·1	11·2	2	·855
751	+51° 136		37·4	51	15	340·3	6·65	9·1	14·0	2	·811
752	+53° 139	0	40·2	53	41	279·1	5·67	9·2	11·6	2	·932

New Double Stars—continued.

No.	B.D.	R.A. 1900. h m	Decl. ° ′	P. °	D. ″	Mags.		Nights.	Date. 1909.	
753	+53° 145	0 42·3	53 32	115·7	8·05	9·1	9·1	2	·932	
754	+53° 180	0 50·3	53 30	194·0	6·35	8·8	9·5	2	·932	
755	+51° 244	1 5·2	51 45	334·8	4·03	9·1	12·0	4	·860	AB
				124·5	3·13	12·2	12·3	3	·856	CD
				182·1	10·20			3	·856	AC
				295·3	47·74	E=	9·3	2	·862	AE
756	+53° 252	6·2	54 7	209·3	3·50	8·5	10·6	4	·875	
757		13·0	51 18	58·6	1·76	9·4	11·2	3	·824	
758	+51° 280	13·4	51 17	247·3	5·05	9·2	10·7	3	·824	AB
				351·4	34·00	C=	9·6	3	·824	AC
759	+51° 334	28·6	51 39	88·8	9·27	8·6	9·0	2	·800	
760	+53° 394	42·9	53 23	298·2	11·26	8·6	9·1	2	·896	
761	+53° 395	43·2	53 25	249·6	4·92	8·5	11·7	2	·896	
762	+51° 418	45·2	51 18	264·1	2·83	10·3	10·8	3	·942	BC
				278·9	60·13	A=	8·5	2	·932	AB
763	+51° 454	1 52·1	51 26	329·9	5·18	9·0	13·7	2	·932	
764	+51° 566	2 18·1	51 26	94·0	2·32	9·2	12·0	2	·807	BC
				62·1	35·67	A=	8·5	2	·807	AB
				219·1	19·67	a=	11·5	2	·807	Aa
765	+52° 617	36·3	52 20	93·8	6·73	8·7	12·3	3	·957	
766	+52° 635	2 44·8	52 49	295·8	4·18	8·6	10·8	3	·881	
767	+51° 685	3 2·8	51 47	129·9	6·37	9·1	9·5	2	·932	
768	+52° 672	8·3	52 23	47·3	2·58	10·2	11·5	4	·894	BC
				314·6	73·47	A=	8·1	2	·861	AB
769	+50° 730	9·1	50 28	138·8	5·85	9·1	11·0	2	·852	
770	+51° 780	3 41·8	51 56	65·4	2·42	9·0	12·0	2	·947	BC
				49·5	70·20	A=	9·0	2	·947	AB
771	+52° 841	4 24·9	52 26	56·0	9·27	8·6	11·7	2	·972	
772	+53° 1141	7 16·5	53 24		1±	9·3	9·3	1	·137	
773	+50° 2115	14 42·2	50 48	83·8	5·67	9·0	12·0	2	·413	
774	+51° 1979	15 3·8	51 23	232·5	3·10	9·0	9·2	2	·413	
775	+51° 1989	15 11·2	51 14	222·1	6·75	9·3	9·4	2	·413	
776	+53° 1934	17 16·0	53 21	314·9	1·90	10·5	10·7	2	·679	BC
				151·4	29·35	A=	8·6	2	·679	AB
777	+52° 2066	30·0	52 40	351·4	5·97	8·1	12·0	4	·597	
778	+50° 2438	34·5	50 44	27·6	13·47	9·0	12·0	2	·614	BC
				104·1	57·62	A=	8·0	2	·614	AB
779	+51° 2262	47 6	51 5	341·8	9·12	8·8	12·0	2	·603	
780	+54° 1929	17 57·8	54 16	155·3	2·98	9·2	11·8	3	·700	

New Double Stars—continued.

No.	B.D.	R.A. 1900. h m	Decl. ° ′	P. °	D. ″	Mags.		Nights.	Date. 1909.	
781	+53° 2019	18 0·5	53 57	277·3	3·50	9·1	11·4	2	·647	
782	+50° 2536	9·2	50 25	107·1	6·32	9·2	13·6	3	·612	
783	+51° 2337	12·1	51 5	239·0	8·27	9·1	14·0	3	·601	
784	+51° 2399	29·5	51 6	65·6	6·47	8·7	9·2	2	·595	
785	+50° 2615	31·1	50 52	95·8	8·55	8·6	9·0	2	·606	AB
				330·7	18·52	C=	12·0	4	·614	AC
786	+50° 2616	31·1	50 39	324·1	5·41	9·2	9·5	4	·616	
787	+50° 2639	37·1	51 1	37·6	7·75	8·5	9·5	2	·679	
788	+51° 2441	49·2	51 8	317·4	3·05	8·7	9·7	4	·623	
789	+51° 2480	18 58·7	51 12	327·5	3·80	9·2	11·7	2	·605	
790	+51° 2518	19 7·0	51 12	106·8	4·14	8·5	11·3	4	·647	
791	+51° 2557	17·4	51 19	312·7	10·35	8·5	11·3	3	·640	
792	+51° 2588	23·6	51 11	252·3	7·48	8·8	10·7	3	·596	
793	+50° 2820	32·5	50 33	233·1	6·54	9·3	11·7	2	·758	BC
				85·9	15·27	a=	13·5	2	·758	Aa
				247·3	43·40	A=	9·0	2	·758	AB
794	+51° 2664	39·5	51 47	119·7	8·32	9·0	13·2	3	·634	BC
				261·2	42·53	A=	8·3	3	·634	AB
795		44·1	50 36	319·4	3·98	9·4	10·1	2	·613	AB
				51·8	14·45	C=	14·0	1	·608	AC
796	+50° 2899	47·9	51 5	357·9	7·07	9·1	11·3	3	·670	
797	+50° 2919	19 51·9	50 21	154·0	3·96	9·3	12·0	4	·600	
798	+50° 2977	20 2·7	50 13	326·7	2·97	9·7	9·9	2	·606	BC
				104·9	80·40	A=	8·5	1	·594	AB
799	+47° 3051	11·1	47 25	9·8	2·45	9·2	9·3	3	·715	AB
				74·0	3·72	9·6	14·0	3	·715	CD
				224·4	32·22			2	·633	AC
800	+50° 3058	17·2	50 57	150·3	2·25	10·0	12·0	2	·662	CD
				103·6	40·27	A=	8·3	2	·662	AC
				314·6	28·25	B=	8·8	2	·662	AB
				96·8	112·65	E=	8·2	2	·662	AE
801	+44° 3454	20·5	45 6	170·8	9·70	9·0	9·6	3	·880	
802	+48° 3129	24·0	48 49	236·9	4·55	9·4	11·0	3	·812	
803	+47° 3114	25·0	47 55	354·4	5·78	8·0	11·2	3	·774	
804		30·9	44 59	122·5	2·41	9·6	10·7	3	·893	BC
				116·3	31·80	A=	9·5	2	·885	AB
805	+45° 3219	31·4	45 10	135·3	8·95	9·3	10·3	2	·885	
806	+47° 3160	20 36·5	47 44	253·9	6·57	8·5	13·9	4	·730	AB
				31·4	10·97	C=	13·2	4	·730	AC

New Double Stars—continued.

No.	B.D.	R.A. 1900. h m	Decl. ° ′	P. °	D. ″	Mags.		Nights.	Date. 1909.	
807	+48° 3185	20 37·3	48 34	155·4	3·27	8·7	11·1	4	·744	
808	+47° 3166	37·7	48 7	155·4	3·45	9·2	10·6	3	·640	
809		38·2	47 39	91·0	2·60	9·4	10·2	2	·631	
810	+47° 3188	44·5	47 29	42·1	18·33	5·5	14·5	3	·631	
811	+48° 3230	48·9	49 0	138·1	1·97	9·1	9·3	2	·855	
812	+47° 3210	49·6	47 51	140·2	6·27	9·2	10·8	2	·821	
813	+47° 3226	52·4	47 32	235·0	5·16	9·1	11·6	5	·649	
814	+47° 3233	53·3	47 31	356·5	4·52	8·7	12·1	4	·652	
815	+47° 3247	20 56·4	47 20	230·6	5·85	9·1	9·6	2	·679	
816	+47° 3283	21 2·1	47 34	34·0	2·60	9·3	9·4	3	·830	
817	+47° 3297	3·8	47 20	352·3	11·43	7·5	12 0	2	·915	
818	+48° 3289	4·3	48 27	80·8	2·35	11·2	12·2	2	·621	BC
				42·7	47·67	A=	8·2	2	·621	AB
819	+48° 3318	10·8	49 6	201·7	4·70	9·5	10·1	4	·807	
820	+48° 3350	17·2	48 57	290·7	3·17	9·7	11·6	2	·674	BC
				59·0	33·42	A=	8·8	2	·674	AB
821	+48° 3382	22·6	48 39	272·2	7·60	9·1	10·3	3	·639	
822	+48° 3385	22·9	48 45	37·2	1·57	9·0	9·2	3	·639	
823	+48° 3386	23·0	48 51	278·4	9·33	8·1	11·7	3	·639	
824	+47° 3434	27·0	47 17	1·7	6·00	9·1	9·8	2	·868	
825	+48° 3457	36·3	48 41	253·1	9·60	7·5	11·8	3	·618	AB
				285·6	55·10	C=	8·4	2	·612	AC
826	+48° 3523	46·8	48 40	127·0	3·56	9·0	11·9	4	·645	
827	+49° 3642	47·4	50 12	243·8	3·38	9·3	9·6	3	·609	
828	+50° 3474	50·1	51 5	89·1	7·15	9·1	9·6	2	·631	
829	+48° 3553	53·0	48 24	284·2	6·25	9·1	12·7	3	·714	
830	+49° 3708	55·8	49 56	302·5	3·85	9·3	11·8	3	·768	
831	+48° 3576	21 57·7	48 52	78·5	6·57	8·7	12·3	3	·814	AB
				310·8	20·69	C=	9·1	2	·821	AC
832	+47° 3693	22 2·7	47 45	132·3	7·60	9·1	13·0	3	·679	
833	+47° 3696	3·9	47 49	259·9	2·60	9·1	9·2	3	·667	
834		15·4	51 12	335·0	2·55	9·3	10·7	2	·631	
835	+50° 3668	15·8	51 10	199·5	4·02	9·0	10·7	2	·702	
836	+47° 3819	26·4	47 57	165·7	3·32	9·1	12·3	2	·773	
837	+49° 3882	27·6	49 34	231·8	5·45	8·4	12·1	3	·756	AB
				231·2	12·43		11·7	3	·756	AC
838	+49° 3887	22 29·0	49 14	250·4	6·18	9·3	12·0	3	·799	AB
				182·7	18·32		11·0	2	·806	AC

New Double Stars—continued.

No.	B.D.	R.A. 1900. h m	Decl. ° ′	P. °	D. ″	Mags.		Nights.	Date. 1909.	
839	+48° 3765	22 29·3	48 18	65·5	4·78	9·9	10·7	3	·854	BC
				101·7	31·09	A=	9·1	2	·852	AB
840		31·1	48 13	29·2	4·38	9·6	13·2	3	·854	
841		31·3	48 13	95·1	2·83	9·8	9·9	3	·854	
842	+47° 3867	34·6	47 24	111·6	3·42	9·1	9·9	2	·878	
843	+47° 3877	35·7	48 13	210·4	8·35	9·1	9·2	2	·859	
844	+48° 3804	36·5	48 21	39·6	4·45	9·2	11·2	2	·859	
845	+47° 3889	37·6	47 43	36·7	4·37	8·8	12·2	2	·878	
846	+46° 3800	38·4	46 59	161·2	5·25	9·0	14·0	2	·904	
847	+46° 3814	40·6	46 26	254·4	9·87	9·1	12·0	3	·842	
848		40·7	50 3	93·8	2·25	9·3	9·5	2	·722	
849	+47° 3938	46·0	47 49	9·7	10·49	8·2	12·5	3	·939	
850	+47° 3946	47·0	47 51	307·3	7·27	8·6	11·3	3	·939	
851	+47° 3949	47·4	47 33	252·9	7·22	8·8	9·5	2	·915	
852	+46° 3856	47·7	47 2	328·2	6·70	8·4	10·0	3	·818	
853	+47° 3957	48·2	48 1	226·1	3·55	9·3	13·7	3	·950	
854	+48° 3869	48·2	48 25	89·0	2·70	9·0	11·5	2	·963	
855	+47° 3969	50·6	48 11	228·4	4·97	9·1	9·7	2	·655	
856	+48° 3918	22 58·8	48 32	5·9	7·37	9·1	11·8	4	·722	
857	+47° 4153	23 21·1	47 28	161·0	3·32	9·2	10·5	2	·790	
858	+47° 4165	23·2	48 2	122·0	3·37	10·1	10·4	4	·814	BC
				237·1	29·38	A=	9·5	4	·814	AB
859	+47° 4216	32·6	47 59	265·1	2·65	9·8	9·9	3	·816	BC
				216·6	87·84	A=	8·0	2	·809	AB
860		36·3	49 29	46·9	2·42	9·5	9·8	3	·931	
861	+48° 4146	40·3	48 32	278·0	5·62	9·2	9·7	3	·701	
862	+48° 4148	40·8	48 23	39·7	6·01	8·6	12·0	2	·725	AB
				227·9	10·32		12·7	2	·725	AC
863	+49° 4242	45·4	49 30	330·9	3·65	9·3	9·7	2	·984	
864	+48° 4239	23 59·7	49 5	173·9	9·02	8·7	10·2	2	·861	

Notes.

768.—There is a 9·8 mag. star nearer than B, and north of a line joining A and B.

772.—Professor Burnham kindly looked up the star, and has sent me the following measures:—

AB	P	91°·9	D	1″·04	9·6	9·7
AC		354·3		6·50		14·0

1909·70.

The *comes* C was not noted by me.

773.—The *f.* and brighter of two stars.

795.—Another *comes* mag. 13·5 more distant in the same direction as C.

806.—The nearer *comes* suspected on several nights to be a Ghost.
810.—The *comes* is about the *minimum visible* of the 17¼-in. with mirrors newly silvered.
817.—A *comes* at P 350°·9 D 24″·0 mag. 12·0, and a 13·5 mag. 85° ± 24″.
825.—A *comes* 14 mag. 20″ ± N of A.
826.—A pair *S.p.* 10·0 12·0 5″ ± and two others, about the same distance apart *f*.
831.—A 12·0 mag. P 131° D 16″, an 11 mag. P 34° D 20″, an 11·5 P 66°·4 D 22″.
837.—An 8·7 mag. P 177°·1 D 46″·9.
843.—Two other pairs in field *p*.

Observations of Minor Planets, from photographs taken with the 30-inch Reflector of the Thompson Equatorial and the Astrographic 13-inch Refractor at the Royal Observatory, Greenwich, during the year 1908.

(Communicated by the Astronomer Royal.)

The following positions were obtained from photographs taken with the 30-inch reflector and the Astrographic 13-inch refractor at the Royal Observatory, Greenwich. They are referred to the equator and equinox of 1908·0, and are corrected for the part of the aberration arising from the Earth's motion. They are not corrected for parallax or for the part of the aberration arising from the planet's motion.

The plates were measured with the astrographic micrometer. Six reference stars were, as a rule, measured with the planets, their positions being derived, when possible, from the catalogues of the Astronomische Gesellschaft, or from the Radcliffe Catalogue 1890.

Log Parallax Correction = log Parallax Factor − log Δ.

Date and G.M.T. 1908.	R.A. 1908·0.	Log Parallax Factor.	Dec. 1908·0.	Log Parallax Factor.	Corr. for Earth's Aberrn. R.A.	Corr. for Earth's Aberrn. Dec.
d h m s	h m s		° ′ ″		s	″
			(26) Proserpina.			
Jan. 3 9 26 39	4 37 25·72	−8·589	+25 9 1·5	+0·589	+1·27	+3·0
4 9 49 16	4 36 43·71	+7·946	25 7 50·7	0·588	1·26	3·0
			(478) Tergeste.			
Jan. 3 9 51 2	4 53 40·52	−8·361	+13 3 21·0	+0·736	+1·24	+0·5
			(37) Fides.			
Jan. 3 10 11 18	5 0 22·65	−7·258	+28 3 34·0	+0·540	+1·38	+2·7
4 11 12 50	4 59 42·56	+8·036	28 1 36·3	0·540	1·37	2·8
			(198) Ampella.			
Jan. 9 9 33 12	6 7 33·97	−9·137	+20 49 39·6	+0·660	+1·40	−0·5
11 9 45 9	6 5 37·93	−9·006	20 43 12·0	0·656	1·38	0·5

Date and G.M.T. 1908.	R.A. 1908·0.	Log Parallax Factor.	Dec. 1908·0.	Log Parallax Factor.	Corr. for Earth's Aberrn. R.A.	Corr. for Earth's Aberrn. Dec.
d h m s	h m s		° ′ ″		s	″
(270) Analita.						
Jan. 9 9 50 20	6 10 34·08	−9·062	+21 46 44·0	+0·645	+1·42	−0·5
11 9 57 2	6 8 27·85	−8·940	21 45 52·5	0·641	1·39	0·5
(47) Aglaia.						
Jan. 9 10 8 47	6 47 3·07	−9·191	+30 9 10·9	+0·526	+1·56	−1·3
11 10 9 6	6 45 6·01	−9·140	30 9 50·6	0·521	1·55	1·1
22 9 46 56	6 35 13·27	−8·917	30 7 21·5	0·509	1·44	0·2
(288) Glauke.						
Jan. 9 10 34 2	7 16 41·89	−9·180	+20 43 14·6	+0·665	+1·45	−2·7
11 10 33 26	7 14 41·58	−9·132	20 50 1·5	0·660	1·45	2·6
(24) Themis.						
Jan. 9 10 52 37	7 26 44·75	−9·145	+23 9 45·9	+0·631	+1·47	−3·0
11 10 50 4	7 24 55·71	−9·108	23 13 40·2	0·628	1·47	2·9
22 10 3 52	7 15 10·38	−9·067	23 32 18·8	0·622	1·43	2·4
(654) Zelinda.						
Feb. 12 9 1 1	6 54 18·60	−8·637	+ 6 22 0·3	+0·793	+1·04	−5·2
(508) Princetonia.						
Apr. 7 10 10 48	10 55 58·40	+8·495	+23 12 3·2	+0·618	+1·13	−2·8
8 10 28 13	10 55 28·45	+8·840	23 9 10·7	0·621	1·10	2·7
(134) Sophrosyne.						
Mar. 19 11 35 6	11 13 22·15	+8·204	+ 2 48 53·8	+0·818	+1·23	−8·0
23 10 19 2	11 9 35·87	−8·864	2 54 46·7	0·817	1·21	7·9
(624) Hector.						
Mar. 23 11 19 0	11 24 55·40	−7·505	− 1 46 25·1	+0·846	+1·23	−8·3
31 11 9 27	11 20 45·43	+8·583	1 34 6·5	0·845	1·17	8·2
Apr. 7 10 45 24	11 17 27·49	+8·694	1 24 7·4	0·844	1·10	8·0
(444) Gyptis.						
Apr. 8 10 8 47	11 39 51·77	−8·583	− 1 27 17·6	+0·850	+1·14	−7·9
May 6 10 8 54	11 29 16·91	+9·177	+ 1 4 51 7	0·830	0·73	5·4
18 10 16 11	11 29 23·96	+9·353	1 37 26·0	0·828	0 51	3·9
20 10 7 15	11 29 41·39	+9·348	1 40 49·7	0·828	0·48	3·6
(46) Hestia.						
Mar. 23 10 41 25	11 53 35·71	−9·030	+ 0 35 51·0	+0·832	+1·25	−8·1
(472) Roma.						
Apr. 3 11 32 31	12 57 9·47	−8·791	+17 56 50·4	+0·686	+1·32	−7·5
7 11 39 51	12 53 47·22	−8·267	18 21 58·6	0·679	1·33	6·8
24 11 24 3	12 40 59·17	+8·949	19 18 40·9	0·672	1·22	3·5

Date and G.M.T. 1908.					R.A. 1908·0.			Log Parallax Factor.	Dec. 1908·0.			Log Parallax Factor.	Corr. for Earth's Aberrn. R.A.	Dec.
	d	h	m	s	h	m	s		°	′	″		s	″
								(19) Fortuna.						
Apr.	3	11	53	45	13	14	50·48	−8·725	− 8	30	34·3	+0·880	+1·27	−7·6
	7	12	0	10	13	11	14·84	−8·111	8	6	48·9	0·878	1·28	7·8
	24	11	43	0	12	56	36·39	+8·958	6	27	31·4	0·871	1·20	7·5
								(578) 1905 R.Z.						
Apr.	8	10	45	50	13	12	55·31	−9·100	− 5	36	11·9	+0·865	+1·27	−7·7
	24	11	57	45	12	58	30·16	+9·043	4	49	25·1	0·861	1·20	7·3
May	4	10	40	44	12	51	0·29	+8·794	4	30	4·9	0·861	1·09	6·7
	6	10	24	29	12	49	43·95	+8·709	4	27	33·4	0·861	1·07	6·6
								(503) Evelyn.						
Apr.	8	11	5	45	13	18	5·75	−9·011	− 1	26	35·3	+0·844	+1·26	−7·7
May	4	10	59	3	12	57	55·15	+8·908	0	18	31·2	0·838	1·11	5·9
	6	10	39	6	12	56	49·33	+8·803	0	17	6·1	0·838	1·08	5·7
								(313) Chaldaea.						
Apr.	8	11	24	48	14	12	54·68	−9·194	− 2	22	27·8	+0·848	+1·23	−7·5
	13	10	43	18	14	8	56·10	−9·259	− 1	26	22·2	0·844	1 26	7·4
May	4	11	17	42	13	51	45·98	+8·407	+ 1	46	40·8	0·825	1·24	5·7
	6	10	49	37	13	50	22·81	−7·681	+ 1	59	24·4	0·823	1·23	5·4
								(579) 1905 S.D.						
Apr.	8	11	38	40	14	17	52·23	−9·156	+ 0	26	12·2	+0·833	+1·23	−7·8
May	6	11	0	55	13	55	45·14	+7·681	1	31	24·5	0·827	1·24	5·5
	18	10	42	17	13	47	37·24	+8·793	1	28	5·3	0·827	1·12	4·1
								(511) Davida.						
May	6	11	29	3	14	44	33·12	−8·460	+ 5	34	40·8	+0·799	+1·31	−5·4
	18	11	0	48	14	35	56·54	+8·206	5	41	40·9	0·798	1·25	3·7
	21	11	6	1	14	33	57·70	+8·667	5	39	59·3	0·798	1·23	3·2
	23	11	13	16	14	32	41·57	+8·856	5	38	5·5	0·799	1·21	2·9
								(8) Flora.						
May	18	11	22	14	15	39	22·30	−8·716	− 11	18	0·4	+0·891	+1 45	−4·6
	23	11	23	27	15	34	0·93	−8·990	11	7	14·5	0·891	1·34	4·4
June	2	11	20	20	15	23	54·53	+8·811	10	52	56·9	0·889	1·28	3·9
	3	10	17	15	15	23	1·00	−8·466	10	52	9·0	0·894	1·27	3·9
								(113) Amalthea.						
May	18	12	2	9	16	5	35·64	−8·492	− 12	24	41·6	+0·896	+1·37	−4·2
	27	11	11	15	15	56	49·01	−8·611	12	12	37·1	0·895	1·36	3·9
June	2	11	40	33	15	51	10·37	+8·743	12	9	12·5	0·894	1·33	3·6
								(581) Tauntonia.						
May	27	11	48	57	16	29	20·88	−8·504	− 2	8	49·0	+0·848	+1·35	−3·3

Date and G.M.T 1908.	R.A. 1908·0.	Log Parallax Factor.	Dec. 1908·0.	Log Parallax Factor.	Corr. for Earth's Aberrn. R.A.	Corr. for Earth's Aberrn. Dec.
d h m s	h m s		° ′ ″		s	″
(79) Eurynome.						
June 18 11 51 54	16 58 2·65	+8·833	−16 56 20·6	+0·910	+1·39	−1·0
19 10 50 37	16 57 8·15	−8·426	16 54 31·7	0·911	1·38	1·2
(57) Mnemosyne.						
June 2 12 20 27	17 19 9·42	−8·351	− 5 2 17·2	+0·864	+1·35	−2·3
10 10 31 57	17 13 13·76	−9·129	4 38 21·1	0·860	1·36	1·5
18 11 28 0	17 7 13·95	+8·108	4 21 41·0	0·861	1·34	0·7
19 11 8 19	17 6 30·88	−8·050	4 20 11·9	0·860	1·34	0·6
(3) Juno.						
July 2 12 11 17	18 49 48·45	+7·807	− 4 44 36·7	+0·862	+1·37	+1·4
(547) Praxedis.						
July 2 12 33 6	19 53 39·54	−8·784	+ 5 35 50·2	+0·798	+1·29	−0·1
Aug. 1 12 21 52	19 29 22·77	+9·161	4 27 37·4	0·809	1·28	+5·5
(17) Thetis.						
Nov. 26 11 59 30	3 1 43·69	+9·103	+ 8 37 17·6	+0·778	+1·27	+4·6
Dec. 9 9 32 7	2 52 12·16	−8·053	8 25 22·0	0·776	1·12	3·5
(92) Undina.						
Nov. 25 10 8 14	3 47 56·72	−9·114	+ 9 44 13·0	+0·770	+1·35	+4·1
26 11 39 33	3 47 3·86	+8·356	9 43 55·5	0·765	1·34	4·0
(407) Arachne.						
Nov. 25 11 1 10	3 48 30·91	−8·713	+30 9 29·3	+0·503	+1·54	+4·5
26 11 11 22	3 47 28·49	−8·412	30 4 1·5	0·503	1·54	4·6
Dec. 9 8 49 46	3 35 22·22	−9·217	28 45 40·6	0·554	1·43	5·6
(120) Lachesis.						
Nov. 25 11 1 10	3 50 19·23	−8·757	30 10 48·8	+0·504	+1·54	+4·5
26 11 11 22	3 49 23·53	−8·470	30 8 17·4	0·502	1·54	4·6
Dec. 9 8 49 46	3 38 12·20	−9·233	29 28 51·8	0·544	1·13	5·6
(106) Dione.						
Nov. 25 11 26 48	4 40 11·44	−8·975	+22 52 39·6	+0·627	+1·45	+2·7
26 12 16 15	4 39 15·03	−7·239	22 52 39·9	0·621	1·46	2·7
Dec. 11 9 18 29	4 25 45·89	−9·252	22 49 13·6	0·645	1·43	3·3
(28) Bellona.						
Nov. 26 12 32 48	6 30 13·12	−9·178	+10 29 47·7	+0·765	+1·17	+1·5
(674) Rachel.						
Dec. 11 8 47 54	1 31 24·46	+8·771	+ 3 28 32·0	+0·814	+0·73	+2·6
15 8 15 56	1 31 13·30	+8·505	3 55 7·1	0·811	0·70	+2·1
Jan. 11 7 10 31	1 40 47·43	+8·920	7 37 1·8	0·784	0·14	−0·2
15 6 51 57	1 43 37·61	+8·877	8 14 27·2	0·779	0·07	−0·5

Royal Observatory, Greenwich:
1910 *January* 14.

Observations of Comet a 1909, *from photographs taken with the 30-inch Reflector of the Thompson Equatorial at the Royal Observatory, Greenwich.*

(*Communicated by the Astronomer Royal.*)

The following positions were obtained from photographs taken with the 30-inch reflector at the Royal Observatory, Greenwich. They are referred to the equator and the equinox of 1909·0, and are corrected for the part of the aberration arising from the Earth's motion. They are not corrected for parallax, or for the part of the aberration arising from the comet's motion.

The plates were measured with the astrographic micrometer. Six reference stars were, as a rule, measured with the comet, their positions being derived, when possible, from the Astronomische Gesellschaft catalogues.

Log Parallax Correction = log Parallax Factor − log Δ.

Date and G.M.T.	R.A. 1909·0.	Log Parallax Factor.	Dec. 1909·0.	Log Parallax Factor.	Corr. for Earth's Aberrn. R.A.	Corr. for Earth's Aberrn. Dec.
	h m s		° ′ ″		s	″
June 22 13 32 24	1 59 44·13	−9·666	+38 11 5·3	+0·760	−0·84	−11·0
30 12 22 51	2 27 16·02	9·697	47 15 29·0	0·793	0·97	12·3
30 12 34 49	2 27 18·02	9·706	47 15 58·6	0·779	0·97	12·3
July 1 11 56 57	2 30 59·46	9·681	48 14 28·7	0·817	1·00	12·4
10 13 6 32	3 8 31·41	9·806	55 59 52·1	0·680	1·23	13·1
19 12 20 26	3 50 52·49	9·843	61 37 45·9	0·726	1·54	13·1
20 12 46 24	3 55 56·82	9·870	62 9 25·2	0·683	1·58	13·1
23 12 51 30	4 11 9·09	9·892	63 35 48·4	0·666	1·70	13·0

Royal Observatory: Greenwich.

Observations of Occultations of Stars by the Moon, made at the Royal Observatory, Greenwich, in the Year 1909.

(*Communicated by the Astronomer Royal.*)

Day, 1909.	Phenomenon.	Telescope.	Mean Solar Time of Observation. h m s	Observer.
Jan. 25	Disapp. 30 Piscium	Astrog. Equat.	5 58 1·11	H. F.
25	,, ,,	Sheeps. ,,	5 58 1·38	W.
25	,, ,,	Astrog. Finder	5 58 1·41	W. S.
25	,, ,,	Sheeps. ,,	5 58 1·28	G. C.
25 (*a*)	,, W.B. XXIII. 1130	Astrog. Equat.	6 6 59·14	W. S.
25	,, W.B. XXIII. 1101	,,	6 9 2·30	H. F.
25	,, 33 Piscium	,,	7 42 5·41	W. S.
Feb. 22	,, Anon.	,,	6 40 27·31	W. S.
Mar. 26	,, B.D. +21°731	,,	10 12 5·05	W. S.
Apr. 3 (*b*)	,, ν Virginis	Sheeps. Equat.	9 54 52·25	W.
3	,, ,,	Astrog. ,,	9 54 52·65	A. L.
May 31	,, κ Virginis	28 inch ,,	8 43 58·86	A. L.
31	,, ,,	Corbett Tel.	8 43 59·16	A. W.
31 (*c*)	,, ,,	Sheeps. Equat.	8 43 58·92	H. A.
31 (*c*)	,, ,,	,, Finder	8 43 58·82	S. D.
July 25 (*e*) (*d*)	Reapp. 2 Libræ	Astrog. Equat.	9 2 20·50	W. S.
Sep. 27	Disapp. B.A.C. 8004	,,	7 45 47·11	W. S.
27 (*f*) (*e*)	Reapp. ,,	,,	8 48 2·60	W. S.
Oct. 5	,, B.D. +24°1365	,,	10 38 58·18	W. S.
5	,, B.D. +24°1386	,,	11 37 22·91	W. S.
5	,, B.D. +24°1401	,,	12 36 5·00	W. S.
5	,, B.D. +25°1454	,,	14 2 12·39	W. S.
6 (*e*)	Disapp. κ Geminorum	,,	11 58 48·27	W. S.
6	Reapp. B.D. +24°1746	,,	12 2 10·71	W. S.
6	,, B.D. +24°1755	,,	12 31 48·34	W. S.
6	,, κ Geminorum	,,	12 33 50·30	W. S.
6 (*h*)	,, B.D. +24°1763	,,	12 53 54·71	W. S.
6	,, B.D. +24°1764	,,	13 0 55·74	W. S.
Nov. 19	Disapp. B.D. −22°5669	,,	6 59 7·94	W. S.
22	,, 30 Piscium	,,	6 32 29·56	H. A.
22	,, 33 ,,	,,	8 26 32·25	S. E.
22	,, ,,	Merz Tel.	8 26 32·59	C. D.
22	,, ,,	Sheeps. Equat.	8 26 32·03	R. C.
22	,, ,,	Corbett Tel.	8 26 31·81	A. Wh.
22 (*e*) (*g*)	Reapp. ,,	,,	9 29 0·01	A. Wh.

Day, 1909.	Phenomenon.	Telescope.	Mean Solar Time of Observation. h m s	Observer.
Dec. 3	Reapp. B.D. +14°2266	Astrog. Equat.	11 48 53·97	W. S.
3	,, B.D. +14°2269	,,	12 6 28·79	W. S.
3	,, B.D. +14°2277	,,	12 55 28·83	W. S.
20	Disapp. B.D. −2°83	,,	6 34 26·20	W. S.
20	,, B.D. −2°93	,,	8 4 23·16	W. S.
24	,, ω^2 Tauri	,,	6 10 43·57	J. H. P.
24	,, B.D. +20°740	,,	8 26 17·23	J. H. P.
24	,, B.D. +20°751	,,	9 5 2·38	J. H. P.
24	,, B.D. +20°748	,,	9 20 45·49	J. H. P.

All the phenomena were at the dark limb except those to which the letter *e* is prefixed.

(*a*) Faint. (*b*) Practically instantaneous. (*c*) Good observation; star bright. (*d*) Definition poor; limb tremulous. (*e*) At bright limb of Moon. (*f*) Observer uncertain if his count of seconds was correct. (*g*) Observer considered that the time noted was possibly 1 second late. (*h*) The star reappeared and then disappeared again for a fraction of a second.

The apertures of the telescopes used are as follows:—

28-inch Equatorial	28 inches.
Corbett telescope (attached to 28-inch)	6·5 ,,
Merz telescope (attached to Thompson Equatorial)	12·8 ,,
Astrographic Equatorial (guiding telescope)	10 ,,
,, ,, Finder	2·8 ,,
Sheepshanks Equatorial	6·7 ,,
,, ,, Finder	2·8 ,,

The initials S. E., C. D., H. F., W., W. S., R. C., S. D., H. A., A. W., G. C., A. L., A. Wh., J. H. P., are those of Mr. Eddington, Mr. Davidson, Mr. Furner, Mr. Witchell, Mr. Stevens, Mr. Cullen, Mr. S. Daniels, Mr. Acton, Mr. Witney, Mr. Cody, Mr. Lyne, Mr. A. Wheate, and Mr. Peirce.

Royal Observatory, Greenwich:
1910 *January* 14.

On the Theory of Aberration and the Principle of Relativity.
By H. C. Plummer, M.A.

1. In a former paper (vol. lxix. p. 496) the theory of aberration has been discussed from the standpoint of ordinary optical theory. This suffices for the conclusion that beyond the astronomical effects as commonly understood no first-order optical effects can be put in evidence, and that, in the absence of greater experimental refinement, it is impossible for the observer to detect his own absolute motion in the ether. With this position the astronomer must be content. If we have reason to be dissatisfied with the results of the efforts hitherto made to determine the constant of aberration, we have little ground for taking into account the second-order effects. But on the other hand an optical experiment has been performed which cannot be reconciled with the ordinary theory, and we have been forced to admit that the theory, if not actually erroneous, can be no more than an approximation to the truth. Meanwhile the electronic theory of matter has been developed, and has embraced in its synthesis an explanation of this difficulty. The result is that the Principle of Relativity, with its far-reaching implications, has obtained a cardinal position in modern science. It is possible that there are some astronomers who are not familiar with the literature of the subject, and to whom an elementary account of the new ideas in physics may be of interest. Accordingly the subject will be approached from the purely optical side, and an attempt will be made to present the theory, which is a product of the last decade and is due chiefly to Professor Lorentz,* in the simplest possible way.

2. According to the principle of relativity, it is impossible to find experimental evidence of the absolute motion through the ether. Let us consider the bearing of this on a very simple experiment. Let AB and AC (fig. 1) be actual lengths l' and l at right angles to one another, and let mirrors be placed at B and C perpendicular to AB, AC, so that rays of light from A will be reflected back to A. Now suppose that the whole apparatus is carried through the ether with the velocity v in the direction AB, the velocity of light being U. The time from A to B and back again will be

$$t_1 = l'/(\mathrm{U} - v) + l'/(\mathrm{U} + v) = 2l'\mathrm{U}/(\mathrm{U}^2 - v^2).$$

But the time by the path AC′A′ in the ether will be

$$t_2 = 2\mathrm{AC}'/\mathrm{U} = 2l/\sqrt{(\mathrm{U}^2 - v^2)}.$$

Hence if $t_1 = t_2$,

$$l' = l\sqrt{(1 - v^2/\mathrm{U}^2)}.$$

* The fundamental hypothesis concerning the contraction of matter in motion is due to Fitzgerald; the first mention of his suggestion occurs in an abstract of a paper by Sir O. Lodge (*Nature*, 1892, June 16). This reference I owe to Professor Whittaker.

The actual lengths l', l must thus be different if the light-times are the same. Nevertheless no difference can be detected between the lengths by any test that we can apply, or the principle of relativity will be contravened. Hence we are led to admit a universal contraction of all material systems in the direction of their motion through the ether, according to the law

$$l' = l \cos \beta, \quad \text{where} \quad \sin \beta = v/\mathrm{U}.$$

The dimensions transverse to the motion are considered unaltered.

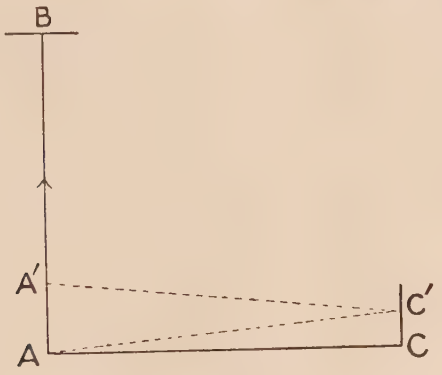


Fig. 1.

If it were objected that the light-times considered above could not be determined with sufficient accuracy to support so astonishing an inference, the objection would apply only to the simple form of argument adopted. In practice the light-times can be compared with extraordinary precision by making the beams interfere on returning to A. In fact we are dealing, as it were, with a schematic representation of the Michelson-Morley experiment, the null result of which is the fundamental fact on the optical side which has to be explained.

3. The uniform contraction in one dimension of the moving material system, the possibility of which we have thus been led to entertain, suffices of itself for the discussion of some interesting optical problems. As an example, the position of the focus of a moving parabolic mirror may be chosen. Let $\mathrm{BA_1B'}$ be the mirror (fig. 2) and $\mathrm{F_1}$ its focus when at rest. Let $\mathrm{BA_2B'}$ be the figure of the mirror when in motion towards a star situated on its axis and $\mathrm{F_2}$ the simultaneous position of the point of the (contracted) apparatus which corresponds to $\mathrm{F_1}$. The incident wave-front is BCB′. If $\mathrm{CF_1}$ is taken as the axis of x and CB as the axis of y, the equation of $\mathrm{BA_1B'}$ will be

$$z^2 + y^2 = 4f(x + b)$$

where $A_1F_1 = f$, $A_1C = b$. If we admit a contraction λ^{-1} which will make $CF_1 = \lambda . CF_2$, the equation of BA_2B' becomes

$$z^2 + y^2 = 4f\lambda(x + b/\lambda).$$

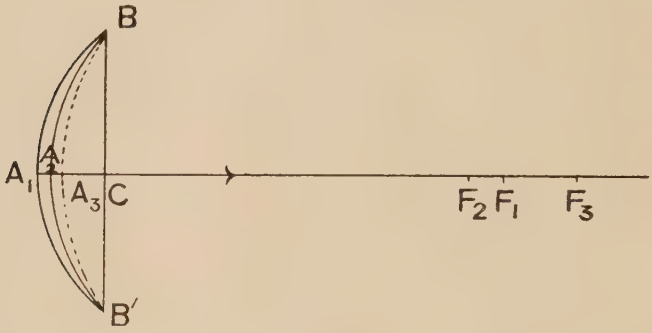


FIG. 2.

This is a real deformation of the figure of the mirror. But if BA_3B' is the surface at which the advancing wave-front actually meets the moving mirror

$$A_2A_3/v = A_3C/U = A_2C/(v + U)$$

where v is the velocity of the mirror, and the virtual surface on which the wave falls becomes

$$z^2 + y^2 = 4f\lambda(1 + v/U)\{x + b/\lambda(1 + v/U)\}.$$

This is still a paraboloid, and the wave BCB′ will reach its focus F_3 after a time

$$t = f\lambda(U + v)/U^2 + b/\lambda(U + v).$$

Hence no change of focus will be detected provided $F_2F_3 = vt$. Now

$$CF_3 = f\lambda(1 + v/U) - b/\lambda(1 + v/U)$$

$$CF_2 = f/\lambda - b/\lambda$$

so that

$$F_2F_3 = f\lambda(1 + v/U) - f/\lambda + bv/\lambda(U + v)$$

which is equal to vt if

$$f\lambda.v(U + v)/U^2 = f\lambda.(U + v)/U - f/\lambda$$

or

$$\lambda^{-2} = (U + v)(1/U - v/U^2) = 1 - v^2/U^2.$$

This is the law of contraction previously obtained, and the result is a complete compensation of the optical effect due to the motion of the mirror.

When a telescope is moving directly towards a star, Veltmann's theorem * shows that the motion will not affect the relative position of the focus to the first order. But a second order effect will remain outstanding, although it will be too small to be ascertained by focal settings. According to our present ideas, however, no effect of any order is to be expected, and our example shows how the compensation operates in a particularly simple case.

4. In what precedes we have contemplated only a change in one dimension of the moving system, or, as it may be expressed, a transformation of one coordinate in space. We have now to consider a related transformation to apparent or, as it is called, "local" time. Let axes be taken attached to the moving system, the measured coordinates being ξ, η, ζ. Let their position at the time $t = 0$ be the axes fixed in space, the corresponding coordinates being x, y, z. The motion of the system is supposed parallel to the axis of x or ξ.

We now imagine a new system of time τ, which depends not only on the absolute time t but also on the position in space. It is sufficient for our purpose to suppose that

$$\tau = at - bx, \quad \xi = cx - dt, \quad \eta = y, \ \zeta = z.$$

All optical phenomena which would be described by an observer at rest in space in terms of x, y, z and t will be described by an observer in motion in terms of ξ, η, ζ and τ. Now one result of the principle of relativity and of the constant velocity of light is that the spherical wave-front

$$x^2 + y^2 + z^2 = \mathrm{U}^2 t^2$$

must appear to the moving observer as

$$\xi^2 + \eta^2 + \zeta^2 = \mathrm{U}^2 \tau^2,$$

for a spherical wave which actually converges to a point in reality must appear to converge to a point and to move with the velocity U. This requires

$$(cx - dt)^2 - \mathrm{U}^2(at - bx)^2 \equiv x^2 - \mathrm{U}^2 t^2$$

or

$$c^2 - b^2\mathrm{U}^2 = 1, \quad a^2 - d^2\mathrm{U}^{-2} = 1, \quad cd = ab\mathrm{U}^2,$$

which can be satisfied by

$$a = c = \sec \beta, \quad b\mathrm{U} = d/\mathrm{U} = \tan \beta.$$

Here β is arbitrary, but we must also have $\xi = 0$ identical with $x = vt$, or

$$v = d/c = \mathrm{U} \sin \beta,$$

* *Mon. Not.*, lxix. p. 504. Professor Whittaker considers that Veltmann's paper added nothing to Fresnel's original paper in *Annales de Chimie*, ix. p. 57 (1818). In attaching importance to Veltmann's theorem I have followed Jamin (*Cours de Physique*).

and thus the complete transformation is determined :

$$\tau = t \sec\beta - x \tan\beta/\mathrm{U}$$
$$\xi = x \sec\beta - \mathrm{U}t \tan\beta$$
$$\eta = y,\ \zeta = z, \quad \sin\beta = v/\mathrm{U}.$$

5. The physical interpretation of this transformation is simple. The coordinates y and z are unaltered, but we have at any given time t

$$\xi_2 - \xi_1 = (x_2 - x_1)\sec\beta.$$

The measured distance between two points in the direction of motion is therefore greater than the actual distance in the ratio of $\sec\beta : 1$. This accords with the idea already entertained that the corresponding dimension of the material system, including any scale which may be used for making measures, is actually diminished in consequence of the motion in the ratio $\cos\beta : 1$.

At a given position in the ether

$$\frac{\partial\tau}{\partial t} = \sec\beta,$$

which means that a stationary clock made to synchronise with passing clocks keeping "local" time must be accelerated in the ratio $\sec\beta : 1$ when compared with the standard time of space t. But on the other hand,

$$\frac{d\tau}{dt} = \sec\beta - v\tan\beta/\mathrm{U} = \cos\beta,$$

which means that a "local" clock moving through the ether with the velocity v has a rate retarded in the ratio $\cos\beta : 1$ when compared with the "standard" clock. The result of the transformation is that we have established a consistent system of apparent time which is such that if we imagine luminous clock-faces at all points of the moving system indicating local time, those which are at equal *measured* distances from a given station will *appear* to show the same time, and this a time differing from the local time of the station by an amount equal to the apparent constant distance divided by U.

6. We now see that *if the laws of optics relative to the moving system, expressed in terms of* ξ, η, ζ *and* τ, *are formally the same as the laws for a stationary system, expressed in terms of* x, y, z *and* t, *there will be no possibility of detecting the fact of motion by any optical experiment made with apparatus which is carried with the system.*

Now we have from the transformation of § 4,

$$\frac{d\xi}{d\tau} = \left(\frac{dx}{dt} - \mathrm{U}\sin\beta\right)\Big/\left(1 - \frac{dx}{dt}\sin\beta/\mathrm{U}\right)$$
$$\frac{d\eta}{d\tau} = \frac{dy}{dt}\cos\beta\Big/\left(1 - \frac{dx}{dt}\sin\beta/\mathrm{U}\right)$$
$$\frac{d\zeta}{d\tau} = \frac{dz}{dt}\cos\beta\Big/\left(1 - \frac{dx}{dt}\sin\beta/\mathrm{U}\right)$$

for the relative velocities. Let W be the absolute velocity of a ray which is parallel to the axis of x or ξ in a moving medium which has the refractive index μ when at rest. Then

$$\frac{d\xi}{d\tau} = (\mathrm{W} - v)/(1 - \mathrm{W}v/\mathrm{U}^2).$$

But this is $\mu^{-1}\mathrm{U}$ if the apparent ray-velocity is unaltered by the motion. Hence

$$\mathrm{W} - v = \mu^{-1}\mathrm{U}(1 - \mathrm{W}v/\mathrm{U}^2)$$

or

$$\mathrm{W} = \mu^{-1}\mathrm{U} + (1 - \mu^{-2})v/(1 + \mu^{-1}v/\mathrm{U}).$$

The second member on the right is the apparent drift of plane light-waves which are normal to the direction of motion. To the first order it agrees with Fresnel's expression $(1 - \mu^{-2})v$ and is consistent with Fizeau's experiment, which cannot be performed accurately enough to verify it to a higher order. For the rest we have a general explanation of the null effect of optical experiments without supposing that the ether is carried along in the neighbourhood of the Earth without relative motion.

7. According to the electromagnetic theory a plane wave of light depends upon two periodic vectors, the components of which contain the factor

$$\sin 2\pi w\{t - (ax + by + cz)/\mathrm{W}\},$$

where a, b, c are the direction cosines of the wave-normal and W is the wave-velocity. To the observer moving with velocity v, the corresponding factor is

$$\sin 2\pi w'\{\tau - (a'\xi + b'\eta + c'\zeta)/\mathrm{W}'\}.$$

The transformation of § 4 gives the relations:

$$\begin{aligned} w &= w'(\sec\beta + a'\mathrm{U}\tan\beta/\mathrm{W}') \\ wa/\mathrm{W} &= w'(a'\sec\beta/\mathrm{W}' + \tan\beta/\mathrm{U}) \\ wb/\mathrm{W} &= w'b'/\mathrm{W}' \\ wc/\mathrm{W} &= w'c'/\mathrm{W}', \end{aligned}$$

or, if the transformation be reversed (which only requires the sign of β to be changed),

$$\begin{aligned} w' &= w\sec\beta(1 - a\mathrm{U}\sin\beta/\mathrm{W}) \\ a'/\mathrm{W}' &= (a/\mathrm{W} - \sin\beta/\mathrm{U})/(1 - a\mathrm{U}\sin\beta/\mathrm{W}) \\ b'/\mathrm{W}' &= b\cos\beta/\mathrm{W}(1 - a\mathrm{U}\sin\beta/\mathrm{W}) \\ c'/\mathrm{W}' &= c\cos\beta/\mathrm{W}(1 - a\mathrm{U}\sin\beta/\mathrm{W}). \end{aligned}$$

The latter equations give

$$1/\mathrm{W}'^2 = \{1 - 2a\mathrm{W}\sin\beta/\mathrm{U} - (1 - a^2 - \mathrm{W}^2/\mathrm{U}^2)\sin^2\beta\} \div \mathrm{W}^2(1 - a\mathrm{U}\sin\beta/\mathrm{W})^2,$$

or

$$1/\mathrm{W}^2 = \{1 + 2a'\mathrm{W}'\sin\beta/\mathrm{U} - (1 - a'^2 - \mathrm{W}'^2/\mathrm{U}^2)\sin^2\beta\} \div \mathrm{W}'^2(1 + a'\mathrm{U}\sin\beta/\mathrm{W}')^2$$

which express the apparent wave-velocity in a moving medium in terms of the absolute, or *vice versa*. For a vacuum we put $W = W' = U$, and we have

$$w' = w \sec\beta(1 - a\sin\beta)$$
$$a' = (a - \sin\beta)/(1 - a\sin\beta)$$
$$b' = b\cos\beta/(1 - a\sin\beta)$$
$$c' = c\cos\beta/(1 - a\sin\beta).$$

8. The laws of stellar aberration and of the Doppler effect are at once deducible, as Einstein * has shown. For a star situated in the direction making an angle ϕ with the direction of motion of the observer we have $a = -\cos\phi$ and $a' = -\cos\phi'$, where ϕ' is the apparent direction. Then we see that

$$w' = w \sec\beta(1 + \sin\beta\cos\phi)$$
$$\cos\phi' = (\cos\phi + \sin\beta)/(1 + \sin\beta\cos\phi).$$

These expressions can, however, be simplified. For the latter equation gives

$$\sin\phi' = \cos\beta\sin\phi/(1 + \sin\beta\cos\phi).$$

Hence

$$w'\sin\phi' = w\sin\phi.$$

Again,

$$\frac{1 - \cos\phi'}{1 + \cos\phi'} = \frac{(1 - \sin\beta)(1 - \cos\phi)}{(1 + \sin\beta)(1 + \cos\phi)}$$

or

$$\tan\tfrac{1}{2}\phi' = \sqrt{\left(\frac{U - v}{U + v}\right)}\tan\tfrac{1}{2}\phi,$$

which puts the law of aberration in an extremely simple form. If we use wave-lengths instead of wave-frequencies we have

$$\lambda'/\sin\phi' = \lambda/\sin\phi.$$

This form, which connects directly the apparent wave-length with the apparent aberrational position of the star, becomes useless when $\phi = \phi' = 0$, but in this case the product of the last two equations gives in the limit

$$\lambda' = \sqrt{\left(\frac{U - v}{U + v}\right)}\lambda,$$

which is the new form of Doppler's principle. It must be remembered that v is the motion of the observer relative to the medium, and that λ depends on the unknown velocity of the source of light relative to the medium. In some cases we may fairly assume that λ is constant, but λ as well as ϕ is originally unknown and, if the principle of relativity be accepted in its widest extent, remains unknowable.

* *Ann. der Physik*, 1905, No. 10, p. 891.

9. The geometrical significance of the new law of aberration is interesting. In the first place we may consider the stereographic projection of the celestial sphere on the tangent plane perpendicular to the direction of motion. The form of the law of aberration

$$\tan \tfrac{1}{2}\phi' = \sqrt{\left(\frac{U-v}{U+v}\right)} \tan \tfrac{1}{2}\phi$$

shows that the effect of aberration is simply to alter the *scale* of the projection. But the stereographic projection is a conformal representation of the sphere. Hence actual configurations on the sphere are only changed conformally by the effect of aberration, or in other words any small area is altered only in size and not in shape. We know that aberration merely changes the scale of a photograph of a small part of the sky, and the truth of this fact now becomes independent of the velocity (however large) of the observer. Also stars which appear to lie on a circle at any one time will continue to do so permanently.

Another geometrical representation is obtained by assimilating ϕ' to the eccentric and ϕ to the true anomaly in an ellipse whose eccentricity is $v/U = \sin \beta$. This means that we view the *apparent* celestial sphere from the centre. Then, to pass from the apparent direction of a star to its actual direction, we must imagine the sphere transformed into an ellipsoid by contracting its dimensions perpendicular to the axis along which motion takes place in the ratio of $\cos \beta : 1$. The true direction will then be inferred by viewing the ellipsoid of revolution from the focus which is in advance of the centre. Conversely, we may interchange ϕ and ϕ' if we employ the other focus. We thus see that if we consider the true celestial sphere to undergo the contraction just specified, the apparent positions of the stars are given by viewing the ellipsoid from that focus which follows the centre.

10. *Secular Aberration.*—In what precedes, the question of all second-order effects connected with aberration has taken a new form. But we may return to the old order of ideas to consider briefly the subject of secular aberration which arises from the motion of the solar system through the ether of space, and which was first discussed by Villarceau.* It is a very simple matter when regarded from the old point of view, but it has led to some apparent confusion. It has been seen (vol. lxix. p. 505) that if the light which leaves a star at the time T is observed at the time t, the observed direction of the star coincides with the actual direction of a fictitious body which is supposed to start from the position of the star at time T, and to continue in motion during the time $(t - T)$ with a velocity equal and parallel to that of the Earth at the time t.

Now the velocity of the Earth is compounded of its velocity v relative to the centre of mass of the solar system and the velocity

* *Conn. des Temps*, 1878.

V (supposed constant) of the system. If V′ is the resultant and U the velocity of light, we have

$$ES = U(t - T), \quad SS' = V'(t - T),$$

E (fig. 3) being the position of the Earth at time t, S the position of the star at time T, ES′ the observed direction, and SS′ the

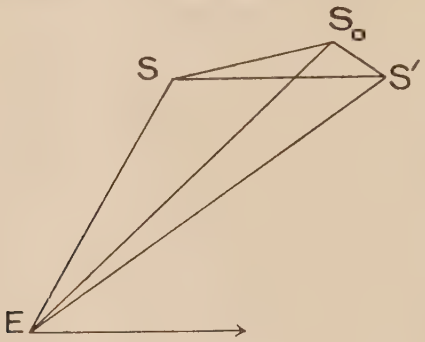


FIG. 3.

virtual displacement of the star. But this displacement is compounded of two, SS_0 and S_0S', such that

$$SS_0 = V(t - T), \quad S_0S' = v(t - T).$$

Let $ES_0 = U\tau$. If then we ignore the velocity V, as in practice we do, and assume a virtual star in the permanently displaced position S_0, we shall infer from observation a fictitious velocity of the Earth v' such that

$$S_0S' = v(t - T) = v'\tau$$

or

$$v'/v = (t - T)/\tau = ES/ES_0 = U/\sqrt{(U^2 + V^2 + 2UV\cos\phi)},$$

where ϕ is the angle between the true direction of the star and the direction of the secular motion of the solar system. Otherwise expressed, the result is as if the constant of aberration for the given star is changed in the ratio of $U : \sqrt{(U^2 + V^2 + 2UV\cos\phi)}$. This is an elementary deduction of Villarceau's result, and on the assumed premises no other effect is to be expected.

The subject of secular aberration was discussed by Seeliger,* who arrived at results which are quite inadmissible. His expressions for the effects of aberration in R.A. and declination become infinite at the pole, whereas it is obvious that there is no singularity in the neighbourhood of this point in the sky. There is no special connection between the pole and the direction of the operative vectors considered above, and such results can only be due to faulty analysis. It would have been unnecessary to refer

* *A.N.*, 2610.

to this paper, had not the same errors been reproduced by Weinek* in an elaborate memoir of recent date. A correct treatment of the question has been given by Schwarzschild † on the lines of the present paragraph. But of course the whole question has now entered on a new phase.

11. It is easy to see that the old and the new laws of aberration agree to the first order of small quantities. Beyond the first order, the present accuracy of astronomical observations does not encourage us to look for any means of discriminating between them. Nevertheless it is interesting to ask whether the principle of relativity, as outlined above, has robbed us of the theoretical possibility of detecting the effect of the motion of the solar system through the ether of space. This has been asserted, and we have now to find a satisfactory justification for the assertion. It may be thought that a sufficient argument is contained in the statement italicised in § 6. But this does not appear to cover the case of aberration without a more critical examination. The motion of the observer does affect his observations, even when expressed in terms of the transformed variables, and we cannot dispense with the necessity of explaining in detail how the expected compensation operates when the observations are corrected in a definite, specified manner. And the question is subtle enough to leave ample room for misapprehension. The most obvious instance of this seems worthy of attention. Let us imagine observations made simultaneously by an observer E on the Earth and an observer S on the Sun (or rather the centre of gravity of the solar system, supposed to be moving with uniform velocity through space). Referred in the usual way to a sphere (fig. 4), let S be the apex of the Sun's ether-velocity V_0, and E the apex of the Earth's ether-velocity V_1. Let P represent the true direction of a star, and P_0, P_1 the positions

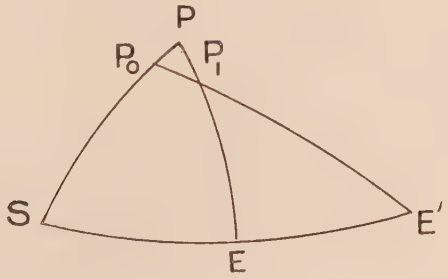


Fig. 4.

which it appears to the observers S and E respectively to occupy. Now we should have complete agreement between the two observers, and total compensation for E, if the correction applied by E had the effect of changing P_1 into P_0. The observer E must

* *Denkschr. der Wiener Akad.*, Bd. 77, p. 204.
† *A.N.*, 3246.

suppose that he is moving relatively to S in the direction of some point E′ on the great circle SE; and if he is to deduce the position P_0 in space from the observed position of the star P_1, the point E′ must lie also on the great circle P_0P_1. Since P_0, P_1, E′ are points on the sides of the triangle SPE, and S, E, E′ are points on the sides of the triangle PP_0P_1, we have

$$\frac{\sin SE'}{\sin EE'} \cdot \frac{\sin EP_1}{\sin P_1P} \cdot \frac{\sin PP_0}{\sin P_0S} = 1$$

$$\frac{\sin P_0E'}{\sin P_1E'} \cdot \frac{\sin P_1E}{\sin PE} \cdot \frac{\sin PS}{\sin P_0S} = 1.$$

Let

$$PS = \phi_0, \quad P_0S = \phi'_0, \quad PP_0 = \phi_0 \quad \phi'_0$$

$$PE = \phi_1, \quad P_1E = \phi'_1, \quad PP_1 = \phi_1 - \phi'_1.$$

Hence

$$\frac{\sin SE'}{\sin EE'} = \frac{\sin(\phi_1 - \phi'_1)}{\sin \phi'_1} \cdot \frac{\sin \phi'_0}{\sin(\phi_0 - \phi'_0)}$$

$$\frac{\sin P_0E'}{\sin P_1E'} = \frac{\sin \phi_1}{\sin \phi'_1} \cdot \frac{\sin \phi'_0}{\sin \phi_0}.$$

The first of these shows that

$$\frac{\sin(\phi_1 - \phi'_1)}{\sin \phi'_1} \div \frac{\sin(\phi_0 - \phi'_0)}{\sin \phi'_0} = \text{const.}$$

for all stars, which happens to be true according to the ordinary theory of aberration. But it is not true according to the "relativity" law of aberration, which gives

$$\frac{\sin(\phi - \phi')}{\sin \phi'} = \frac{\sin \beta + (1 - \cos \beta)\cos \phi}{\cos \beta}.$$

In fact it appears that a law of aberration, which would explain the absence of a visible effect arising from the secular motion by supposing that the corrected observation coincides *in space* with the standard observation, would require simultaneously

$$\sin(\phi - \phi')/\sin \phi' = f_1(V), \quad \sin(\phi - \phi')/\sin \phi = f_2(V),$$

and this is not possible. We have then to look in another direction for the explanation of the way in which the "relativity" law effects a compensation, and for the necessary hint I am indebted to Mr. Eddington, who very kindly supplied me with a particular case which was both simple and illuminating.

12. A general proof will now be given. Let the ether-velocity of the observer S be $V_0 = U \sin \beta_0$, and of the observer E be $V_1 = U \sin \beta_1$, and let the angle between them be α, being measured from V_0 towards V_1. The components of V_0 along and per-

pendicular to V_1 are $V_0 \cos\alpha, -V_0 \sin\alpha$. Hence for E the components of the apparent velocity of S are (by § 6)

$$\frac{V_0 \cos\alpha - U \sin\beta_1}{1 - V_0 \cos\alpha \sin\beta_1/U}, \qquad \frac{-V_0 \sin\alpha \cos\beta_1}{1 - V_0 \cos\alpha \sin\beta_1/U}$$

or

$$\frac{U(\cos\alpha \sin\beta_0 - \sin\beta_1)}{1 - \cos\alpha \sin\beta_0 \sin\beta_1}, \qquad \frac{-U \sin\alpha \sin\beta_0 \cos\beta_1}{1 - \cos\alpha \sin\beta_0 \sin\beta_1}.$$

Now E must infer that his own velocity relative to S, $V_{10} = U \sin\beta_{10}$, has these components reversed in sign; and if θ be the angle between V_1 and the resultant we have

$$\sin\beta_{10} \cos\theta = (\sin\beta_1 - \cos\alpha \sin\beta_0)/(1 - \cos\alpha \sin\beta_0 \sin\beta_1)$$
$$\sin\beta_{10} \sin\theta = \sin\alpha \sin\beta_0 \cos\beta_1/(1 - \cos\alpha \sin\beta_0 \sin\beta_1)$$

and we deduce

$$\cos\beta_{10} = \cos\beta_0 \cos\beta_1/(1 - \cos\alpha \sin\beta_0 \sin\beta_1).$$

We take the z-axis perpendicular to V_0 and V_1 throughout, and

(i) The x-axis parallel to V_0. If (λ, μ, ν) are the direction cosines of a star in its true position, S observes this star in the direction (by § 7)

$$\lambda_0 = \frac{\lambda + \sin\beta_0}{1 + \lambda \sin\beta_0}, \qquad \mu_0 = \frac{\mu \cos\beta_0}{1 + \lambda \sin\beta_0}, \qquad \nu_0 = \frac{\nu \cos\beta_0}{1 + \lambda \sin\beta_0}.$$

(ii) We turn the x-axis through the angle α to bring it into the direction of V_1. The direction cosines of the star in its true position become

$$\lambda \cos\alpha + \mu \sin\alpha, \quad \mu \cos\alpha - \lambda \sin\alpha, \quad \nu,$$

and E will observe it in the direction

$$\lambda_1 = (\lambda \cos\alpha + \mu \sin\alpha + \sin\beta_1)/\{1 + (\lambda \cos\alpha + \mu \sin\alpha) \sin\beta_1\}$$
$$\mu_1 = (\mu \cos\alpha - \lambda \sin\alpha) \cos\beta_1/\{1 + (\lambda \cos\alpha + \mu \sin\alpha) \sin\beta_1\}$$
$$\nu_1 = \nu \cos\beta_1/\{1 + (\lambda \cos\alpha + \mu \sin\alpha) \sin\beta_1\}.$$

(iii) We turn the x-axis through the further angle θ to bring it into the direction of V_{10}. The direction cosines of the star in its apparent position become

$$\lambda_1 \cos\theta + \mu_1 \sin\theta, \quad \mu_1 \cos\theta - \lambda_1 \sin\theta, \quad \nu_1,$$

and when E has corrected this position for his observed relative velocity V_{10} he will infer that the star lies in the direction

$$\lambda_{10} = (\lambda_1 \cos\theta + \mu_1 \sin\theta - \sin\beta_{10})/\{1 - (\lambda_1 \cos\theta + \mu_1 \sin\theta) \sin\beta_{10}\}$$
$$\mu_{10} = (\mu_1 \cos\theta - \lambda_1 \sin\theta) \cos\beta_{10}/\{1 - (\lambda_1 \cos\theta + \mu_1 \sin\theta) \sin\beta_{10}\}$$
$$\nu_{10} = \nu_1 \cos\beta_{10}/\{1 - (\lambda_1 \cos\theta + \mu_1 \sin\theta) \sin\beta_{10}\}.$$

These have to be compared with λ_0, μ_0, ν_0, remembering that the axes have been turned through an angle $\alpha + \theta$.

13. The process of eliminating λ_1, μ_1, ν_1 and θ is perfectly simple and straightforward, if rather complicated. We find

$$1-(\lambda_1\cos\theta+\mu_1\sin\theta)\sin\beta_{10} = \frac{(1+\lambda\sin\beta_0)\cos^2\beta_1}{\{1+(\lambda\cos\alpha+\mu\sin\alpha)\sin\beta_1\}(1-\cos\alpha\sin\beta_0\sin\beta_1)}$$

whence

$$\nu_{10}=\nu\cos\beta_0/(1+\lambda\sin\beta_0).$$

Also

$$\sin\beta_{10}(\mu_1\cos\theta-\lambda_1\sin\theta)\{1+(\lambda\cos\alpha+\mu\sin\alpha)\sin\beta_1\}(1-\cos\alpha\sin\beta_0\sin\beta_1) = \cos\beta_1\{\mu(\cos\alpha\sin\beta_1-\sin\beta_0)-(\lambda+\sin\beta_0)\sin\alpha\sin\beta_1\}$$

whence

$$\mu_{10}\sin\beta_{10}=\frac{\cos\beta_0\{\mu(\cos\alpha\sin\beta_1-\sin\beta_0)-(\lambda+\sin\beta_0)\sin\alpha\sin\beta_1\}}{(1+\lambda\sin\beta_0)(1-\cos\alpha\sin\beta_0\sin\beta_1)}.$$

Similarly a little reduction gives

$$\lambda_{10}\sin\beta_{10}=\frac{(\lambda+\sin\beta_0)(\cos\alpha\sin\beta_1-\sin\beta_0)+\mu\sin\alpha\cos^2\beta_0\sin\beta_1}{(1+\lambda\sin\beta_0)(1-\cos\alpha\sin\beta_0\sin\beta_1)}.$$

We come now to the interpretation of these expressions. The components of V_1 along and perpendicular to V_0 are $V_1\cos\alpha$, $V_1\sin\alpha$. Hence the apparent components of the relative velocity of E as observed by S are

$$\frac{V_1\cos\alpha-U\sin\beta_0}{1-V_1\cos\alpha\sin\beta_0/U}, \qquad \frac{V_1\sin\alpha\cos\beta_0}{1-V_1\cos\alpha\sin\beta_0/U},$$

or

$$\frac{U(\cos\alpha\sin\beta_1-\sin\beta_0)}{1-\cos\alpha\sin\beta_0\sin\beta_1}, \qquad \frac{U\sin\alpha\cos\beta_0\sin\beta_1}{1-\cos\alpha\sin\beta_0\sin\beta_1}.$$

Hence if χ be the angle between V_0 and the resultant $V_{01}=U\sin\beta_{01}$, we have

$$\sin\beta_{01}\cos\chi=(\cos\alpha\sin\beta_1-\sin\beta_0)/(1-\cos\alpha\sin\beta_0\sin\beta_1)$$
$$\sin\beta_{01}\sin\chi=\sin\alpha\cos\beta_0\sin\beta_1/(1-\cos\alpha\sin\beta_0\sin\beta_1)$$

and we deduce

$$\cos\beta_{01}=\cos\beta_0\cos\beta_1/(1-\cos\alpha\sin\beta_0\sin\beta_1).$$

We see that $\beta_{01}=\beta_{10}$, and when we introduce λ_0, μ_0, ν_0 and χ into the last expressions for λ_{10}, μ_{10}, ν_{10} we obtain simply

$$\lambda_{10}=\lambda_0\cos\chi+\mu_0\sin\chi, \quad \mu_{10}=\mu_0\cos\chi-\lambda_0\sin\chi, \quad \nu_{10}=\nu_0.$$

The interpretation is that the sky as seen by the observer E and referred to the direction in which he observes himself to be moving relative to S, is identical with the sky as seen by the observer S, and referred to the direction in which he observes E to be moving relative to himself. Thus the observations of E,

corrected for his observed relative velocity, give the same configuration as the observations of S uncorrected, and the latter is a permanent standard so long as his velocity is uniform. But for E the whole universe is rotated in *space* through an angle $\theta + \alpha - \chi$. This rotation is precisely that which is required to enable the two observers to identify the same star as occupying the position which each observer imagines to represent the apex of the motion of E relative to S, and thus it could not be detected even though the two observers were in communication. The difficulty discussed in § 11 arises from the fact that the observed velocity of E relative to S and the observed velocity of S relative to E are not in the same straight line. We find that

$$\tan(\chi - \theta) = \frac{\sin\alpha(\cos\beta_0 + \cos\beta_1)}{\cos\alpha(1 + \cos\beta_0\cos\beta_1) - \sin\beta_0\sin\beta_1},$$

which can be reduced to the form

$$\tan\tfrac{1}{2}(\chi - \theta) = \frac{\cos\frac{1}{2}(\beta_1 - \beta_0)}{\cos\frac{1}{2}(\beta_1 + \beta_0)}\tan\tfrac{1}{2}\alpha.$$

Thus $\chi - \theta$ is not equal to α, as it would be in ordinary kinematics.

14. We have thus given a perfectly precise meaning to the application of the principle of relativity to stellar aberration. In doing so we have excluded everything which does not belong to the purely optical aspect of the problem. But stellar aberration is not a purely optical problem. For in practice we do not actually observe the apparent motion of the Sun and use the result to correct our observed positions of the stars. The motion which we do use is derived by calculation from the theory of gravitation. Hence, if we are to be consistent, we must regard Keplerian motion as an appearance, not as a reality. And here we come in contact with the general problem of the dynamics of the electron, which in the historical sense is responsible for the introduction of the principle of relativity. In this wider problem there is a profound modification of the notion of mass, in addition to the transformation which we have admitted in the time and the spatial coordinates. The result of the work of Lorentz and others is to show that these transformations suffice to explain the complete compensation of effects arising from the motion of any system through space over the whole field of electrodynamics as well as of optics. The same will be true of gravitation if gravity can be expressed in terms of electrodynamic entities. If, on the other hand, the nature of gravity were essentially not electrodynamic, it would be possible for some deviation from the accepted laws of dynamics, owing to a motion of translation through space, to be revealed by direct astronomical observations. The possibility of such deviations and of the compensations, partial or complete, which may accompany them, places some astronomical problems in a new position. Thus Poincaré has recently reconsidered* the

* *C.R.*, cxl. p. 1504.

problem of Laplace concerning the finite propagation of gravitation, and has concluded, contrary to Laplace's negative result, that the propagation of gravitation with the velocity of light can be brought into consistency with the facts of observation owing to the partial compensations which are now introduced into the theory. Such questions are in the highest degree subtle and intricate as well as interesting. What it is desired to point out is that the developments of modern physical theory concern the astronomer no less than the physicist.

The Principal Formulæ of Interpolation and Mechanical Differentiation and Integration. By H. C. Plummer, M.A.

1. The difficulty of finding simple and general proofs of the formulæ in most common use by the computer suggests that it may be useful to give them in a connected form. The difference table is often given in something like the following shape :—

$$\begin{array}{l|l|l} f^{-2}(a-w) & & \\ & f^{-1}(a-\tfrac{1}{2}w) & \\ f^{-2}(a) & & f^{0}(a-w) \\ & f^{-1}(a+\tfrac{1}{2}w) & \\ f^{-2}(a+w) & & f^{0}(a) \\ & & f^{0}(a+w) \end{array} \quad \begin{array}{l|l} & f^{2}(a-w) \\ f^{1}(a-\tfrac{1}{2}w) & \\ & f^{2}(a) \\ f^{1}(a+\tfrac{1}{2}w) & \\ & f^{2}(a+w) \end{array} \quad \begin{array}{l} \\ \ldots \\ \\ \ldots \\ \\ \end{array}$$

The tabulated function appears between the vertical lines, with the differences to the right; to the left are the numbers of which the derivation is defined by the fact that each column is followed on the right by its first differences. In addition, the means between two consecutive numbers in any column are indicated by the functional form of the column and the arithmetic mean of the arguments involved, *e.g.* —

$$\tfrac{1}{2}\{f^{1}(a-\tfrac{1}{2}w)+f^{1}(a+\tfrac{1}{2}w)\}=f^{1}(a).$$

This notation is of course unscientific, because, for example, $f^{0}(a+\frac{1}{2}w)$ does *not* indicate the result of substituting $a+\frac{1}{2}w$ for a in $f^{0}(a)$. But it is useful enough for expressing results, for which purpose it is only necessary to indicate a number by the column and line which it occupies in the difference table.

2. In the process of obtaining the results, however, we wish to use the operators of finite differences. In connection with central differences, with which we are chiefly concerned, it is convenient to use a pair of operators. Such a notation in the algebra of difference tables was introduced by the late Professor

Everett.* By using two operators, Δ and δ, the table of differences is given this form:

$\Delta^{-3}\delta$		$\Delta^{-2}\delta^2$		$\Delta^{-1}\delta^3$		δ^4	
	$\Delta^{-2}\delta$		$\Delta^{-1}\delta^2$		δ^3		
Δ^{-2}		$\Delta^{-1}\delta$		δ^2		$\Delta\delta^3$	
	Δ^{-1}		δ		$\Delta\delta^2$		
$\Delta^{-1}\delta^{-1}$		1		$\Delta\delta$		$\Delta^2\delta^2$	
	δ^{-1}		Δ		$\Delta^2\delta$		
δ^{-2}		$\Delta\delta^{-1}$		Δ^2		$\Delta^3\delta$	
	$\Delta\delta^{-2}$		$\Delta^2\delta^{-1}$		Δ^3		
$\Delta\delta^{-3}$		$\Delta^2\delta^{-2}$		$\Delta^3\delta^{-1}$		Δ^4	

The tabulated function is represented symbolically between the vertical lines, and the formation of the table is self-evident. The operator Δ signifies advancing differences in the usual way, while the associated operator δ signifies receding differences. The two operators, which are easily seen to obey the ordinary laws of algebra, are clearly connected by the relation

$$\Delta\delta = \Delta - \delta \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The successive differences of the argument x have been taken as w. Then if D be written for d/dx, we have, by Taylor's theorem,

$$\Delta\delta^{-1} = 1 + \Delta = \exp.\, w\mathrm{D} \quad . \quad . \quad . \quad (2)$$

The algebraic manipulation of the three operators, Δ, δ, and D will yield all the formulæ which are commonly used in interpolation and mechanical quadrature.

1. *Interpolation.*

3. The central differences, as commonly understood, in our table are

$$\tfrac{1}{2}(\Delta + \delta), \quad \Delta\delta, \quad \tfrac{1}{2}(\Delta + \delta)\Delta\delta, \quad (\Delta\delta)^2, \ . \ . \ . \ .$$

The form of these terms suggests the introduction of the operators

$$k = \tfrac{1}{2}(\Delta + \delta),\ \mathrm{K} = \Delta\delta = \Delta - \delta \quad . \quad . \quad . \quad (3)$$

so that

$$\Delta = k + \tfrac{1}{2}\mathrm{K},\ 4k^2 = 4\mathrm{K} + \mathrm{K}^2 \quad . \quad . \quad . \quad (4)$$

Then Newton's formula of interpolation in advancing differences

$$(\Delta\delta^{-1})^n = (1 + \Delta)^n = 1 + \Sigma\binom{n}{r}\Delta^r \quad . \quad . \quad . \quad (5)$$

becomes

$$\begin{aligned}(1 + \Delta)^n &= (1 + k + \tfrac{1}{2}\mathrm{K})^n \\ &= k.u_n + v_n \quad . \quad . \quad . \quad . \quad (6)\end{aligned}$$

* *Quarterly Journal of Mathematics*, xxxi. p. 357, xxxii. p. 306.

where

$$u_n = \binom{n}{1}(1+\tfrac{1}{2}\mathrm{K})^{n-1} + \binom{n}{3}(1+\tfrac{1}{2}\mathrm{K})^{n-3}(\mathrm{K}+\tfrac{1}{4}\mathrm{K}^2) + \cdots$$

$$v_n = (1+\tfrac{1}{2}\mathrm{K})^n + \binom{n}{2}(1+\tfrac{1}{2}\mathrm{K})^{n-2}(\mathrm{K}+\tfrac{1}{4}\mathrm{K}^2) + \cdots$$

It is easy to verify that

$$u_n(1+\tfrac{1}{2}\mathrm{K}) + v_n = u_{n+1}$$
$$u_n(\mathrm{K}+\tfrac{1}{4}\mathrm{K}^2) + v_n(1+\tfrac{1}{2}\mathrm{K}) = v_{n+1}$$
$$\frac{dv_n}{d\mathrm{K}} = \tfrac{1}{2}nu_n .$$

Hence we may write

$$v_n = 1 + n\Sigma b_r \mathrm{K}^r, \quad u_n = n + 2\Sigma(r+1)b_{r+1}\mathrm{K}^r.$$

Substituting these series in the recurrence formulæ, and equating the coefficients of K^{r-1} in the first and of K^r in the second, we have

$$2rb_r + (r-1)b_{r-1} + nb_{r-1} = 2rb_r'$$
$$2rb_r + \tfrac{1}{2}(r-1)b_{r-1} + nb_r + \tfrac{1}{2}nb_{r-1} = (n+1)b_r',$$

where $2rb_r'$ is the coefficient of K^{r-1} in u_{n+1}. Hence

$$(n+1)\{2rb_r + (n+r-1)b_{r-1}\} = r\{2(n+2r)b_r + (n+r-1)b_{r-1}\}$$

or

$$(n+r-1)(n-r+1)b_{r-1} = 2r(2r-1)b_r,$$

which shows that

$$b_r = \binom{n+r-1}{2r-1}\frac{\mathrm{A}}{2r}$$

where A is a constant, which is shown by the terms in K to be unity. Hence

$$v_n = 1 + \tfrac{1}{2}n\Sigma\binom{n+r-1}{2r-1}\frac{\mathrm{K}^r}{r} \quad . \quad . \quad . \quad (7)$$

$$u_n = n + \Sigma\binom{n+r}{2r+1}\mathrm{K}^r \quad . \quad . \quad . \quad . \quad (8)$$

The series in v_n gives the terms involving the even central differences, while ku_n gives the terms involving the odd differences. Expressed in the notation of § 1, the first terms of the complete formula are thus

$$f(a+nw) = f^0(a) + nf^1(a) + \frac{n^2}{2!}f^2(a) + \frac{n(n^2-1)}{3!}f^3(a)$$
$$+ \frac{n^2(n^2-1)}{4!}f^4(a) + \frac{n(n^2-1)(n^2-4)}{5!}f^5(a) + \cdots \quad (9)$$

This formula is known by the name of Stirling, but it might more properly be called the Newton-Stirling formula, since it was first discovered by Newton.

4. The formula of Gauss is easily deduced. Here the tabular differences employed are the even central and the odd of the line below, viz.

$$\Delta, \quad \Delta\delta, \quad \Delta.\Delta\delta, \quad (\Delta\delta)^2, \quad \Delta(\Delta\delta)^2, \ldots,$$

or, as they may be written in terms of k and K,

$$k+\tfrac{1}{2}\mathrm{K}, \quad \mathrm{K}, \quad (k+\tfrac{1}{2}\mathrm{K})\mathrm{K}, \quad \mathrm{K}^2, \quad (k+\tfrac{1}{2}\mathrm{K})\mathrm{K}^2, \ldots.$$

Hence in the place of (6) we have

$$(1+\Delta)^n = (k+\tfrac{1}{2}\mathrm{K})u_n + v_n - \tfrac{1}{2}\mathrm{K}u_n,$$

where, by (7) and (8),

$$v_n - \tfrac{1}{2}\mathrm{K}u_n = \mathrm{V}_n = 1 + \tfrac{1}{2}\Sigma\binom{n+r-1}{2r-1}\binom{n}{r} - 1\Big)\mathrm{K}^r$$

$$= 1 + \Sigma\binom{n+r-1}{2r}\mathrm{K}^r. \qquad (10)$$

This gives the coefficients of the even central differences, while the coefficients of the odd differences of the line below are still given by (8). Expressed in the notation of § 1 the first terms of the formula of Gauss are thus

$$f(a+nw) = f^0(a) + nf^1(a+\tfrac{1}{2}w) + \frac{n(n-1)}{2!}f^2(a) + \frac{n(n^2-1)}{3!}f^3(a+\tfrac{1}{2}w)$$

$$+ \frac{n(n^2-1)(n-2)}{4!}f^4(a) + \frac{n(n^2-1)(n^2-4)}{5!}f^5(a+\tfrac{1}{2}w) + \cdots \qquad (11)$$

The effect of inverting the order of the difference table is to change the sign of w and of the odd differences. Hence similarly

$$f(a-nw) = f^0(a) - nf^1(a-\tfrac{1}{2}w) + \frac{n(n-1)}{2!}f^2(a) - \cdots \qquad (12)$$

By proper choice between (11) and (12) n can always be taken between 0 and $\frac{1}{2}$, and these formulæ are very accurate and convenient when single interpolations have to be made.

5. Bessel's formula contains the odd differences in the line below the central function and the means of the even differences in the adjacent lines. The corresponding operators are

$$\Delta, \quad \tfrac{1}{2}\Delta(\Delta+\delta), \quad \Delta^2\delta, \quad \tfrac{1}{2}\Delta^2\delta(\Delta+\delta), \quad \Delta^3\delta^2, \ldots$$

or, as they may be written,

$$\Delta, \quad (1+\tfrac{1}{2}\Delta)\mathrm{K}, \quad \Delta\mathrm{K}, \quad (1+\tfrac{1}{2}\Delta)\mathrm{K}^2, \quad \Delta\mathrm{K}^2, \ldots$$

since by (3)

$$\tfrac{1}{2}\Delta(\Delta+\delta) = \tfrac{1}{2}\Delta(\mathrm{K}+2\delta) = \tfrac{1}{2}\Delta\mathrm{K} + \mathrm{K}.$$

Now the odd differences are the same as in the formula of Gauss, and we have

$$\begin{aligned}(1+\Delta)^n &= \Delta u_n + \mathrm{V}_n \\ &= \Delta(u_n - \tfrac{1}{2}\mathrm{V}_n) + (1+\tfrac{1}{2}\Delta)\mathrm{V}_n.\end{aligned}$$

But, by (8) and (10),

$$\begin{aligned}u_n - \tfrac{1}{2}\mathrm{V}_n = \mathrm{U}_n &= n - \tfrac{1}{2} + \Sigma\left\{\binom{n+r}{2r+1} - \tfrac{1}{2}\binom{n+r-1}{2r}\right\}\mathrm{K}^r \\ &= (n-\tfrac{1}{2})\left\{1 + \Sigma\binom{n+r-1}{2r}\frac{\mathrm{K}^r}{2r+1}\right\} \quad . \quad . \quad (13)\end{aligned}$$

This gives the coefficients of the differences of odd order, while the coefficients of the mean differences of even order are given by (10). But it must be noticed that the first operator on the function is now $1+\frac{1}{2}\Delta$ instead of 1 as before. Hence we have the formula expressed in the notation of § 1,

$$\begin{aligned}f(a+nw) = f^0(a+\tfrac{1}{2}w) &+ (n-\tfrac{1}{2})f^1(a+\tfrac{1}{2}w) + \frac{n(n-1)}{2!}f^2(a+\tfrac{1}{2}w) \\ &+ (n-\tfrac{1}{2})\frac{n(n-1)}{3!}f^3(a+\tfrac{1}{2}w) + \frac{n(n^2-1)(n-2)}{4!}f^4(a+\tfrac{1}{2}w) \\ &+ (n-\tfrac{1}{2})\frac{n(n^2-1)(n-2)}{5!}f^5(a+\tfrac{1}{2}w) + \cdots \quad . \quad . \quad (14)\end{aligned}$$

The first two terms are often written in a different form, viz.

$$f(a+nw) = f^0(a) + nf^1(a+\tfrac{1}{2}w) + \frac{n(n-1)}{2!}f^2(a+\tfrac{1}{2}w) + \cdots \quad (15)$$

This form, which is due to Bessel, is of course equivalent, but it is not symmetrical with respect to the middle of the interval. The symmetry of (14) is made clearer by writing $m+\frac{1}{2}$ for n. Then

$$\begin{aligned}f(a+mw) = f^0(a+\tfrac{1}{2}w) &+ mf^1(a+\tfrac{1}{2}w) + \frac{m^2-\frac{1}{4}}{2!}f^2(a+\tfrac{1}{2}w) \\ &+ \frac{m(m^2-\frac{1}{4})}{3!}f^3(a+\tfrac{1}{2}w) + \frac{(m^2-\frac{1}{4})(m^2-\frac{9}{4})}{4!}f^4(a+\tfrac{1}{2}w) \\ &+ \frac{m(m^2-\frac{1}{4})(m^2-\frac{9}{4})}{5!}f^5(a+\tfrac{1}{2}w) + \cdots \quad . \quad . \quad . \quad (16)\end{aligned}$$

Here the terms of even order are unchanged when the sign of m is reversed, while the terms of odd order are simply changed in sign. The formula can thus be computed in two parts, and thus serve for two interpolations, corresponding to $\pm m$, at the same time. This is of great advantage in systematic interpolation to regular fractions of the tabular interval, *e.g.* in reducing the 12-hourly places of the Moon to an hourly ephemeris. When $n=\frac{1}{2}$ or $m=0$ we have simply

$$\begin{aligned}f(a+\tfrac{1}{2}w) = f^0(a+\tfrac{1}{2}w) &- \tfrac{1}{8}f^2(a+\tfrac{1}{2}w) + \tfrac{3}{128}f^4(a+\tfrac{1}{2}w) \\ &- \tfrac{5}{1024}f^6(a+\tfrac{1}{2}w) + \cdots \quad . \quad . \quad . \quad (17)\end{aligned}$$

II. *Mechanical Differentiation.*

6. The equation (2) can be written

$$\begin{aligned} w\mathrm{D} &= \log(1+\Delta) \\ &= \Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \cdots \quad . \quad . \quad . \quad (18) \end{aligned}$$

the usual equation in advancing differences. Similarly

$$\begin{aligned} w^2\mathrm{D}^2 &= \{\log(1+\Delta)\}^2 \\ &= \Delta^2 - \Delta^3 + \tfrac{11}{12}\Delta^4 - \tfrac{5}{6}\Delta^5 + \tfrac{137}{180}\Delta^6 - \cdots \quad . \quad (19) \end{aligned}$$

The magnitude of the coefficients is considerable, and these formulæ are only trustworthy in simple cases. They apply only to the tabulated value of the argument. But to every formula of interpolation corresponds a method of interpolation, since

$$x = a + nw, \quad wd/dx = d/dn.$$

Thus from Bessel's formula (14) we derive

$$\begin{aligned} wf'(x) = {} & f^1(a+\tfrac{1}{2}w) + (n-\tfrac{1}{2})f^2(a+\tfrac{1}{2}w) + \tfrac{1}{12}(6n^2-6n+1)f^3(a+\tfrac{1}{2}w) \\ & + \tfrac{1}{12}(2n^3-3n^2-n+1)f^4(a+\tfrac{1}{2}w) \\ & + \tfrac{1}{120}(5n^4-10n^3+5n-1)f^5(a+\tfrac{1}{2}w) + \cdots \quad . \quad (20) \end{aligned}$$

$$\begin{aligned} w^2f''(x) = {} & f^2(a+\tfrac{1}{2}w) + \tfrac{1}{2}(2n-1)f^3(a+\tfrac{1}{2}w) \\ & + \tfrac{1}{12}(6n^2-6n-1)f^4(a+\tfrac{1}{2}w) + \tfrac{1}{24}(4n^3-6n^2+1)f^5(a+\tfrac{1}{2}w) + \cdots (21) \end{aligned}$$

and analogous forms may be derived from (9) and (11).

7. But some particular cases are of special simplicity and importance. When $n=0$, d/dn is the coefficient of n and d^2/dn^2 is twice the coefficient of n^2. Now v_n given by (7) is an even function of n, while u_n given by (8) is an odd function. Hence by means of (6) we have

$$w\mathrm{D} = k\left\{1 - \frac{1^2}{3!}\mathrm{K} + \frac{1^2.\,2^2}{5!}\mathrm{K}^2 - \frac{1^2.\,2^2.\,3^2}{7!}\mathrm{K}^3 + \cdots\right\} \quad . \quad (22)$$

$$\tfrac{1}{2}w^2\mathrm{D}^2 = \frac{\mathrm{K}}{2!} - \frac{1^2}{4!}\mathrm{K}^2 + \frac{1^2.\,2^2}{6!}\mathrm{K}^3 - \frac{1^2.\,2^2.\,3^2}{8!}\mathrm{K}^4 + \cdots \quad . \quad (23)$$

or in the notation of § 1,

$$wf'(a) = f^1(a) - \tfrac{1}{6}f^3(a) + \tfrac{1}{30}f^5(a) - \tfrac{1}{140}f^7(a) + \cdots \quad (24)$$

$$w^2f''(a) = f^2(a) - \tfrac{1}{12}f^4(a) + \tfrac{1}{90}f^6(a) - \tfrac{1}{560}f^8(a) + \cdots \quad (25)$$

Again, if we express V_n by (10) and U_n by (13) in terms of $m = n - \frac{1}{2}$ we get series which are respectively even and odd functions of m, the law of the coefficients being indicated in (16). The coefficient of m in $\Delta\mathrm{U}_n$ is therefore d/dn when $n=\frac{1}{2}$, while the coefficient of m^2 in $(1+\frac{1}{2}\Delta)\mathrm{V}_n$ is $\frac{1}{2}d^2/dn^2$ for the same value of n, since

$$(1+\Delta)^n = \Delta\mathrm{U}_n + (1+\tfrac{1}{2}\Delta)\mathrm{V}_n.$$

Hence

$$wD(1+\Delta)^{\frac{1}{2}} = \Delta\left\{1 - \frac{1^2}{3!}\cdot\frac{K}{4} + \frac{1^2\cdot 3^2}{5!}\left(\frac{K}{4}\right)^2 - \frac{1^2\cdot 3^2\cdot 5^2}{7!}\left(\frac{K}{4}\right)^3 + \cdots\right\} \quad . \quad . \quad (26)$$

$$\tfrac{1}{2}w^2D^2(1+\Delta)^{\frac{1}{2}} = (1+\tfrac{1}{2}\Delta)\left\{\frac{1}{2!}K - \frac{9}{4}\left(\frac{1}{1}+\frac{1}{9}\right)\frac{1}{4!}K^2 + \frac{9}{4}\cdot\frac{25}{4}\left(\frac{1}{1}+\frac{1}{9}+\frac{1}{25}\right)\frac{1}{6!}K^3 - \frac{9}{4}\cdot\frac{25}{4}\cdot\frac{49}{4}\left(\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}\right)\frac{1}{8!}K^4 + \cdots\right\} \quad . \quad (27)$$

the effect of the commutative operator $(1+\Delta)^{\frac{1}{2}}$ being to apply the differentiation at the middle of the tabular interval. In the notation of § 1 these become

$$wf'(a+\tfrac{1}{2}w) = f^1(a+\tfrac{1}{2}w) - \tfrac{1}{6}\cdot\tfrac{1}{4}f^3(a+\tfrac{1}{2}w) + \tfrac{3}{40}\cdot\tfrac{1}{4^2}f^5(a+\tfrac{1}{2}w) - \tfrac{5}{112}\cdot\tfrac{1}{4^3}f^7(a+\tfrac{1}{2}w) + \cdots \quad . \quad (28)$$

$$w^2f''(a+\tfrac{1}{2}w) = f^2(a+\tfrac{1}{2}w) - \tfrac{5}{6}\cdot\tfrac{1}{4}f^4(a+\tfrac{1}{2}w) + \tfrac{259}{360}\cdot\tfrac{1}{4^2}f^6(a+\tfrac{1}{2}w) - \tfrac{32299}{50400}\cdot\tfrac{1}{4^3}f^8(a+\tfrac{1}{2}w) + \cdots \quad . \quad (29)$$

III. *Mechanical Integration.*

8. The usual formulæ of mechanical integration can be deduced immediately from the equations (22), (23), (26), and (27). It is convenient to notice at once that by (4)

$$k^{-1} = k/k^2 = k\mathrm{K}^{-1}(1+\tfrac{1}{4}\mathrm{K})^{-1},$$

that by (3)

$$(1+\Delta)/\Delta = \delta^{-1} = \Delta\mathrm{K}^{-1},$$

and that from the latter

$$\frac{1+\Delta}{1+\frac{1}{2}\Delta} = \frac{1+\frac{1}{2}\Delta}{1+\frac{1}{4}\Delta^2/(1+\Delta)} = \frac{1+\frac{1}{2}\Delta}{1+\frac{1}{4}\mathrm{K}}.$$

Hence (22) gives

$$\begin{aligned} w^{-1}\mathrm{D}^{-1} &= k\mathrm{K}^{-1}(1+\tfrac{1}{4}\mathrm{K})^{-1}(1-\tfrac{1}{6}\mathrm{K}+\tfrac{1}{30}\mathrm{K}^2-\tfrac{1}{140}\mathrm{K}^3+\cdots)^{-1} \\ &= k\mathrm{K}^{-1}(1+\tfrac{1}{12}\mathrm{K}-\tfrac{1}{120}\mathrm{K}^2+\tfrac{1}{840}\mathrm{K}^3-\cdots)^{-1} \\ &= k(\mathrm{K}^{-1}-\tfrac{1}{12}+\tfrac{11}{720}\mathrm{K}-\tfrac{191}{60480}\mathrm{K}^2+\cdots) \quad . \quad . \quad (30) \end{aligned}$$

Similarly (23) gives

$$\begin{aligned} w^{-2}\mathrm{D}^{-2} &= \mathrm{K}^{-1}(1-\tfrac{1}{12}\mathrm{K}+\tfrac{1}{90}\mathrm{K}^2-\tfrac{1}{560}\mathrm{K}^3+\cdots)^{-1} \\ &= \mathrm{K}^{-1}+\tfrac{1}{12}-\tfrac{1}{240}\mathrm{K}+\tfrac{31}{60480}\mathrm{K}^2-\cdots \quad . \quad . \quad (31) \end{aligned}$$

Again, dividing $(1+\Delta)$ by both sides of (26) we obtain

$$w^{-1}D^{-1}(1+\Delta)^{\frac{1}{2}} = \Delta K^{-1}\left\{1 - \frac{1}{6}\cdot\frac{K}{4} + \frac{3}{40}\cdot\left(\frac{K}{4}\right)^2 - \frac{5}{112}\left(\frac{K}{4}\right)^3 \cdots\right\}^{-1}$$

$$= \Delta\left(K^{-1} + \frac{1}{24} - \frac{17}{5760}K + \frac{367}{967680}K^2 - \cdots\right) \quad (32)$$

while similar treatment of (27) gives

$$w^{-2}D^{-2}(1+\Delta)^{\frac{1}{2}} = \frac{1+\frac{1}{2}\Delta}{1+\frac{1}{4}K}\left\{K - \frac{5}{6}\cdot\frac{K^2}{4} + \frac{259}{360}\cdot\frac{K^3}{16} - \frac{3229}{5040}\cdot\frac{K^4}{64} + \cdots\right\}^{-1}$$

$$= (1+\tfrac{1}{2}\Delta)K^{-1}\left\{1 + \frac{1}{6}\cdot\frac{K}{4} - \frac{41}{360}\left(\frac{K}{4}\right)^2 + \frac{397}{5040}\left(\frac{K}{4}\right)^3 - \cdots\right\}^{-1}$$

$$= (1+\tfrac{1}{2}\Delta)K^{-1}\left\{1 - \frac{1}{24}K + \frac{17}{1920}K^2 - \frac{367}{193536}K^3 + \cdots\right\} \quad (33)$$

The coefficients in these resulting series can be determined by simple division.

9. The meaning of the operations represented by the series (30) to (33) is that their application to the tabulated function produces numbers which, when differentiated once or twice as the case may be, return to the original function. They therefore give the first and second integrals, and the results may be expressed in the notation of §1 thus:

$$w^{-1}\int_{a+iw} f(x)dx = f^{-1}(a+iw) - \tfrac{1}{12}f^1(a+iw) + \tfrac{11}{720}f^3(a+iw) - \tfrac{191}{604800}f^5(a+iw) + \cdots \quad (34)$$

$$w^{-2}\iint_{a+iw} f(x)dx = f^{-2}(a+iw) + \tfrac{1}{12}f^0(a+iw) - \tfrac{1}{240}f^2(a+iw) + \tfrac{31}{604800}f^4(a+iw) - \cdots \quad (35)$$

$$w^{-1}\int_{a+i'w} f(x)dx = f^{-1}(a+i'w) + \tfrac{1}{24}f^1(a+i'w) - \tfrac{17}{5760}f^3(a+i'w) + \tfrac{367}{967680}f^5(a+i'w) - \cdots \quad (36)$$

$$w^{-2}\iint_{a+i'w} f(x)dx = f^{-2}(a+i'w) - \tfrac{1}{24}f^0(a+i'w) + \tfrac{17}{1920}f^2(a+i'w) - \tfrac{367}{1935360}f^4(a+i'w) + \cdots \quad (37)$$

These formulæ may be applied to any tabular interval, so that i may be given any integral value, while i' has been written for brevity in the place of $i+\frac{1}{2}$.

10. The upper limit of these integrals is definite, but the lower indefinite. The numbers in the columns of the difference table to the left of the function also contain arbitrary constants, and these can be now determined so as to make the integrals vanish at the lower limit, whether this be taken to be a or $a-\frac{1}{2}w$. If the lower limit be a, (34) shows that the arbitrary constant in the first summation column must be chosen so that

$$f^{-1}(a) = \tfrac{1}{12}f^1(a) - \tfrac{11}{720}f^3(a) + \tfrac{191}{604800}f^5(a) - \cdots$$

Now $f^{-1}(a)$ does not occur actually in the column. But we have

$$f^0(a) = f^{-1}(a + \tfrac{1}{2}w) - f^{-1}(a - \tfrac{1}{2}w)$$
$$f^{-1}(a) = \tfrac{1}{2}\{f^{-1}(a + \tfrac{1}{2}w) + f^{-1}(a - \tfrac{1}{2}w)\} = f^{-1}(a - \tfrac{1}{2}w) + \tfrac{1}{2}f^0(a).$$

Hence we must make

$$f^{-1}(a - \tfrac{1}{2}w) = -\tfrac{1}{2}f^0(a) + \tfrac{1}{12}f^1(a) - \tfrac{11}{720}f^3(a) + \tfrac{191}{60480}f^5(a) - \cdots \quad (38)$$

If the second integral is to vanish at the limit a, (35) shows that

$$f^{-2}(a) = -\tfrac{1}{12}f^0(a) + \tfrac{1}{240}f^2(a) - \tfrac{31}{60480}f^4(a) + \cdots \quad . \quad (39)$$

But the lower limit may also be taken midway between two values of the argument. In this case (36) shows that the first arbitrary constant must be such that

$$f^{-1}(a - \tfrac{1}{2}w) = -\tfrac{1}{24}f^1(a - \tfrac{1}{2}w) + \tfrac{17}{5760}f^3(a - \tfrac{1}{2}w) - \tfrac{367}{967680}f^5(a - \tfrac{1}{2}w) + \cdots \quad . \quad (40)$$

while, according to (37), the second arbitrary constant must be such that

$$f^{-2}(a - \tfrac{1}{2}w) = \tfrac{1}{24}f^0(a - \tfrac{1}{2}w) - \tfrac{17}{1920}f^2(a - \tfrac{1}{2}w) + \tfrac{367}{1935360}f^4(a - \tfrac{1}{2}w) - \cdots$$

In terms of numbers actually occurring in the table this may be written

$$2f^{-2}(a) - f^{-1}(a - \tfrac{1}{2}w) = \tfrac{1}{24}\{f^0(a - w) + f^0(a)\} - \tfrac{17}{1920}\{f^2(a - w) + f^2(a)\} + \tfrac{367}{1935360}\{f^4(a - w) + f^4(a)\} - \cdots$$

But by (40)

$$f^{-1}(a - \tfrac{1}{2}w) = -\tfrac{1}{24}\{f^0(a) - f^0(a - w)\} + \tfrac{17}{5760}\{f^2(a) - f^2(a - w)\} - \tfrac{367}{967680}\{f^4(a) - f^4(a - w)\} + \cdots$$

Hence, from the last two equations,

$$f^{-2}(a) = \tfrac{1}{24}f^0(a - w) - \tfrac{17}{5760}\{2f^2(a - w) + f^2(a)\} + \tfrac{367}{967680}\{3f^4(a - w) + 2f^4(a)\} \cdots \quad . \quad (41)$$

When one number is fixed in each column successively to the left of the function in the difference table, the whole column becomes definite. This is done by the equations just formed. Thus if we prepare the table in accordance with (38) and (39), the lower limit of our integrals will be a. If on the other hand we use (40) and (41), the lower limit will be $a - \frac{1}{2}w$.

11. The object has been to draw attention to the effectiveness of the methods introduced by Everett, and to lay stress on the fundamental importance of the operator K, here explicitly employed, in the development of formulæ in central differences. With this

purpose a simple but systematic development of the most practically useful formulæ in interpolation and quadrature has been given; and in order that practical considerations should be kept in view, proofs of all the "gebräuchlichste Formeln" of Bauschinger's *Tafeln zur Theoretischen Astronomie* (pp. 136, 137) have been provided. Extraneous matter which might have served to illustrate the flexibility of the general method has been excluded. But one point may be noticed in conclusion. The independent derivation of the forms (32) and (33)—or (36) and (37)—has thrown no light on the relation which obviously subsists between them. Now (33) may be written

$$\{\log(1+\Delta)\}^{-2}(1+\tfrac{1}{4}\mathrm{K})^{-\frac{1}{2}} = \mathrm{K}^{-1} - \tfrac{1}{24} + \tfrac{17}{5760}\cdot 3\mathrm{K} - \tfrac{367}{967680}\cdot 5\mathrm{K}^2 + \cdots$$

while (32) may be written

$$\{\log(1+\Delta)\}^{-1} = \mathrm{K}^{-\frac{1}{2}} + \tfrac{1}{24}\mathrm{K}^{\frac{1}{2}} - \tfrac{17}{5760}\cdot \mathrm{K}^{3/2} + \tfrac{367}{967680}\cdot \mathrm{K}^{5/2} - \cdots$$

It is then easy to see that the former will be reproduced when the latter is simply differentiated with respect to $-\mathrm{K}^{\frac{1}{2}}$. For

$$-\frac{d}{d\mathrm{K}^{\frac{1}{2}}}\left\{\log(1+\Delta)\right\}^{-1} = \left\{\log(1+\Delta)\right\}^{-2}\cdot\frac{1}{1+\Delta}\cdot\frac{d\Delta}{d\mathrm{K}^{\frac{1}{2}}}$$

$$= \left\{\log(1+\Delta)\right\}^{-2}\cdot(1+\tfrac{1}{4}\mathrm{K})^{-\frac{1}{2}}$$

since we have

$$\mathrm{K}^{\frac{1}{2}} = \Delta(1+\Delta)^{-\frac{1}{2}},\ d\mathrm{K}^{\frac{1}{2}}/d\Delta = (1+\tfrac{1}{2}\Delta)(1+\Delta)^{-3/2} = (1+\tfrac{1}{4}\mathrm{K})^{\frac{1}{2}}(1+\Delta)^{-1},$$

and this proves the relation connecting the series which occur in the first and second integrals to the middle of a tabular interval.

Errata in Mr. Innes' paper, Observations of Jupiter's Galilean Satellites, 1909.

The table on page 42 requires the following corrections in terms of page xiii of the *Nautical Almanac* for 1910.

II. T.I. Apr. 27 *should be* +1·3; May 22, +2·0; June 23, +1·8; July 25, +2·4. II. T.E. May 15 *should be* −2·5; May 22, −3·4; June 23, −2·3; July 18, −2·2.